T COURSE IN
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TWENTIETH CENTURY TEXT-BOOKS

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A FIRST COURSE IN ELEMENTARY ALGEBRA

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PREFACE

The aim of this volume is to unite those proposals of the present widespread movement for the better teaching of Algebra, which have found general approval, with the Elementary Algebra that has successfully undergone the test of many years' use in the class room.

The subject-matter conforms to the standard courses of study. The topics of the first sixteen chapters have been selected and treated so as to constitute a homogeneous and rounded whole fitted for the first year's work. The remaining four chapters are inserted to meet the requirements of the maximum courses of study. The essentials of the subject, as presented in the main text, are in harmony with the best thought of the times, and are based upon a minimum of mathematical theory and a fuller recognition of the utility of the subject.

Supplementary work has been added to many of the chapters. This consists of additional exercises upon which the teacher may draw and of additional topics required in the maximum courses.

Throughout the treatment the authors have constantly kept in mind both the logical value and the practical utility of the subject.

The logical value of Algebra is of prime importance; hence, the proofs of processes are based upon reasons both correct and satisfying to the mind of the pupil. On the other hand, subtle distinctions and arguments savoring of higher mathematical methods without their true rigor have been avoided.

The utility of Algebra is given the emphasis which it so richly deserves. This is done by making the equation prominent, by introducing simple mensurational and physical formulas, and by applying Algebra to modern industrial, commercial, and scientific problems whose content can readily be under-

stood by the pupil. Useless puzzles and problems relating to past conditions have been excluded, with the exception of a few supplementary problems retained on account of their historical interest.

Graphical work is not presented in the form of selected topics of analytic geometry, but is treated simply and concretely as occasion arises for it in the algebraic work. It is thus made an essential factor in the development and interpretation of Algebra instead of an added difficulty.

The pedagogic treatment is inductive. A few preparatory questions precede the definitions and principles. The presentation of the latter is followed by simple Oral Exercises enforcing and applying in the most effective way the particular facts to be taught. These in turn are followed by Written Exercises carefully graded as to difficulty, which supply the necessary drill in calculation.

The Summaries and Reviews at the ends of chapters furnish systematically, and in small compass, the essentials of Algebra, by reference to which the pupil can best review and unify his knowledge of the subject.

THE AUTHORS.

CONTENTS

ELEMENTARY ALGEBRA

CHAPTER I

LITERAL NOTATION AND ITS USES

1. Numbers represented by Letters. In arithmetic, numbers are represented by means of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. But letters also may be used to stand for numbers.

For example:

p may stand for the number of *pounds* in the weight of a body;

d may stand for the number of *dollars* in a sum of money;

l may stand for the number of units in the *length* of an object, and the like.

2. The use of letters to represent numbers enables us to write statements in very brief form. This is an important feature of algebra.

For example:

1. The length of a lot diminished by $\frac{1}{3}$ of its length is 60 feet.

Using l for the number of feet in the length of the lot, this statement may be written:

l minus $\frac{1}{3}l$ is 60,

or, $l - \frac{1}{3}l = 60.$

2. A man's weight when increased by $\frac{1}{4}$ of itself is 200 lb.

Using w for the number of pounds in the man's weight, this statement may be written:

w plus $\frac{1}{4}w$ is 200,

or, $w + \frac{1}{4}w = 200.$

ORAL EXERCISES

1. If l represents the number of yards in the length of a street, what stands for the length of a street 75 yd. longer?

2. If w represents the number of rods in the width of a farm, what represents the width of a farm 20 rd. narrower?

3. One bank contains d dollars and another 3 times as many. How many dollars in the second bank? How many in both banks?

4. There are n pupils in a class and the same number increased by 13 in another. How many pupils in the second class? In both classes?

5. A merchant invested s dollars and lost $\frac{1}{10}$ of this the first year. How much had he left?

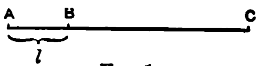


FIG. 1.

6. The line BC in Fig. 1 is 3 times as long as AB. If AB is l units long, how long is BC? AC?

7. In Fig. 2, BC is 4 times as long as AB, and CD is twice as long as AB. If d denotes the length of AB, what is the length of BC? Of CD? Of AD?

FIG. 2.

8. In the following statements c stands for cost, s stands for selling price, and g for gain. Read each statement in words:

1. $s - c = g$. 2. $c + g = s$. 3. $s - g = c$.

3. The Use of Signs. The signs $+$, $-$, $=$, $\times$, $\div$, and $\sqrt{\ }$, have the same meaning in algebra as in arithmetic. But in algebra multiplication is indicated also by the *absence of a sign of operation*. When a sign is needed, the **dot** is often used in preference to the symbol $\times$, which is likely to be mistaken for the letter x.

For example:

a plus b is written $a + b$, just as 3 plus 2 is written $3 + 2$.

a minus b is written $a - b$, just as 3 minus 2 is written $3 - 2$.

a divided by b is written $a \div b$ or $\frac{a}{b}$, just as 3 divided by 2 is written $3 \div 2$ or $\frac{3}{2}$.

The square root of a is written $\sqrt{a}$; the cube root of a, $\sqrt[3]{a}$; and so on.

a times b is written ab, 2 times a is written $2\,a$, and 2 times a plus b times c is written $2\,a + bc$.

2 times 5 is written 2×5 or $2 \cdot 5$.

4. Algebraic Symbols. Letters and other characters used as notations in algebra are called **algebraic symbols.**

5. Algebraic Expressions. Any expression representing a number by use of algebraic symbols is called an **algebraic expression.**

For example: $3b$ and $2a - bc + d$ are algebraic expressions.

The term *literal expression* is often used to denote an algebraic expression involving letters.

6. Value of Algebraic Expressions. The number represented by an algebraic expression is called its **value.**

Accordingly, instead of saying "a represents the number 7," we may say "the value of a is 7," and we shall frequently say for brevity merely "a is 7," or "a equals 7." When any algebraic expression is called a number, its value is meant.

ORAL EXERCISES

1. What is the value of $2n$ when n is 1? When n is 2? When n is 5?

2. What number is $\frac{1}{2}n$ when n is 2? When n is 6? When n is 10?

3. State the value of $2n$ when n is $\frac{1}{2}$. Also when n equals each of the following: $\frac{3}{8}$; $7\frac{1}{2}$; 10; .5; 1.5; 50.

4. State the value of $n+1$ when n equals each of the following: 1; 2; 6; 5; $\frac{1}{2}$; .5; $8\frac{1}{2}$; 100; 0.

5. The length (l) of a box is twice its breadth (b). l is (?) b.

6. How many dimes in 5 dollars? In d dollars?

7. At x cents a peck, what is the cost of a bushel?

8. How many ounces are there in 5 lb.? In x pounds?

9. If x is the number of units in the length of a line, how many units are in a line twice as long? In a line that is half as long?

10. A pair of gloves costs c cents. What would the cost be if the price were raised 5 cents?

11. There were b books on a shelf and 2 were taken down. How many remained on the shelf?

12. A person is x years of age now. How old will he be a year hence? 5 years hence? How old was he three years ago? y years ago?

13. At b cents an ounce, what is the cost of 3 oz.? Of 8 oz.? Of 1 lb.? Of z ounces?

14. Goods that cost c dollars were sold at a profit of s dollars. What was the selling price?

15. A merchant sold goods at 8% above cost; let c be the number of dollars in the cost; then, the gain was 8% of c or $.08\,c$. What was the selling price?

16. Some goods costing x dollars were sold at a gain of 150%. State the selling price.

17. There are x dollars in one bank and $1000 more than twice as much in another. How many dollars in the second bank?

WRITTEN EXERCISES

1. Write the sum of b and c, using the sign of addition.

2. Indicate the subtraction of c from b by using the sign of subtraction.

3. Write the product of a and b as it is expressed in algebra.

4. Write the product of 3 and b and c as it is expressed in algebra.

5. Indicate that a is to be divided by b by using the fractional form.

6. Indicate that the sum of a and b is to be divided by c.

7. Find the value of $2a+1$ when a equals each of the following: 4; 7; $\frac{7}{2}$; $11\frac{1}{2}$; 15; .5; 1.5; 100; 0.

Find the value of $a+b$, when a and b indicate in turn the following numbers:

8. $a=2,\ b=1$.
9. $a=14,\ b=6$.
10. $a=12,\ b=9$.
11. $a=\frac{1}{2},\ b=\frac{1}{4}$.
12. $a=.8,\ b=.5$.
13. $a=1\frac{3}{4},\ b=\frac{5}{8}$.

For each pair of values of a and b above, find the corresponding value of:

14. $a - b$. **15.** ab. **16.** $\frac{a}{b}$. **17.** $\frac{a+b}{ab}$.

18. Draw a rectangle; write b for its base and a for its altitude. Express the area of the rectangle. Its perimeter (sum of its four sides).

19. A rectangular bin is a ft. long, b ft. wide, and c ft. deep. How many cubic feet does it contain?

20. What number does $100a + 10b + c$ represent when $a = 1$, $b = 2$, $c = 3$? When $a = 5$, $b = 4$, $c = 7$?

7. Tabulation of Values. In recording corresponding values it is convenient to use a table like that adjoining.

a	$2a+1$
5	11
0	1
3	7
$8\frac{1}{2}$	18
3.5	8

The values of a are written in the column under a, and the corresponding values of $2a + 1$ are written opposite in the second column. The table records, for example, that when a is 5, $2a + 1$ is 11; that is, $2 \times 5 + 1$. Verify the other values.

WRITTEN EXERCISES

Copy the following tables and supply the numbers to fill the blanks:

1.

n	$3n$
0	( )
1	( )
4	( )
5	( )
$4\frac{1}{3}$	( )
$\frac{10}{3}$	( )

2.

n	$2n-3$
2	( )
$2\frac{1}{2}$	( )
3	( )
$\frac{5}{2}$	( )
1.5	( )

3.

n	$n-1$
1	( )
2	( )
7	( )
$6\frac{1}{2}$	( )
18	( )
25	( )

4.

a	$\frac{1}{2}a+1$
2	( )
12	( )
25	( )
1.8	( )
4.6	( )

5.

v	$10v$
.1	( )
1.2	( )
1.5	( )
6.3	( )
40.4	( )
.05	( )

6.

w	$\frac{3}{4}w$
0	( )
1	( )
24	( )
64	( )
100	( )

7.

a	b	$a+b$
4	5	( )
$6\frac{1}{2}$	$2\frac{1}{2}$	( )
1.8	9.2	( )
2.5	8.3	( )

8.

v	t	vt
$17\frac{1}{2}$	$6\frac{1}{2}$	( )
1.8	.7	( )
4	3.8	( )
12	$7\frac{1}{3}$	( )

9.

l	b	t	lbt
6	8	3	( )
4	5	6	( )
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	( )
$\frac{2}{3}$	$\frac{3}{5}$	.8	( )

SUMMARY

I. **Definitions.**

1. An *algebraic expression* is an expression that represents a number by means of symbols used in algebra. Sec. 5.

2. The *value* of an algebraic expression is the number represented by it. Sec. 6.

II. **Notations.**

1. Letters may be used to stand for numbers. Sec. 1.

2. By using letters to represent numbers many statements may be written in brief form. Sec. 2.

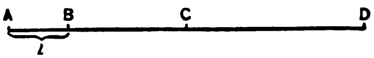
3. The signs $+$, $-$, $=$, $\times$, $\cdot$, $\div$, $\sqrt{\ }$, have the same meaning in algebra as in arithmetic. Sec. 3.

4. Multiplication in algebra is usually indicated by the absence of signs; but when a sign is needed, it is preferable to use the dot. Sec. 3.

REVIEW

ORAL EXERCISES

1. A man invests d dollars in real estate, and 5 times as much in government bonds. How much does he invest altogether?



2. In the figure BC is twice as long as AB, and CD is three times as long as AB. If l denotes the length of AB, what denotes the length of BC? Of CD? Of the entire line?

Find the value of $A+B$, when A and B have the following values:

3. $A=7$, $B=12$.

4. $A=\frac{2}{3}$, $B=\frac{1}{6}$.

5. $A=3x$, $B=4x$.

6. $A=5y$, $B=7y$.

7. Goods that cost d dollars were sold at a profit of p dollars. What was the selling price?

8. In Exercise 7 if d equals \$1 and p equals \$.25, what is the selling price?

WRITTEN EXERCISES

1. Copy, and fill the blanks:

		(1)	(2)	(3)	(4)	(5)
Given	$n =$	6	$\frac{2}{3}$	$\frac{3}{2}$	100	1.3
	$t =$	4	$\frac{1}{3}$	$\frac{5}{2}$	10	20.
Find	$3n - 1 =$	—	—	—	—	—
	$n + t =$	—	—	—	—	—
	$2nt =$	—	—	—	—	—

2. The length of one line is x and that of another is y. How long is the line formed by placing them end to end?

3. Line x is longer than line y. Express the difference between their lengths.

4. What number does $\frac{a}{b}$ represent when $a = 29$, and $b = 58$? When $a = 35$, and $b = 70$? When $a = 127$, $b = 210$?

CHAPTER II

THE EQUATION

8. Preparatory.

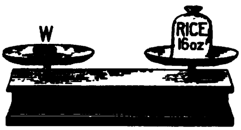


Fig. 1.

1. What weight, w, will balance the package of rice in the figure? What must w be to balance two such packages?

2. If half of the rice be taken from the package, what must w be?

3. What must w be to balance a 16-ounce package and an 8-ounce package?

4. What must w be to balance the pans in each of the following figures?

Fig. 2.

Fig. 3.

Fig. 4.

5. In Fig. 2 the fact that the weights balance is expressed by $w + 3 = 7$. Express the condition that the weights balance in Fig. 3. In Fig. 4.

9. The Equation. If two expressions represent the same number, their equality may be indicated by the sign $=$; such a statement of equality is called an **equation.**

Thus, $w + 3 = 7$, $2w = 8$, $y = 3 + 4$, and $2x + 1 = 9$ are equations.

10. Members of an Equation. The two expressions connected by the sign of equality are called the **members** of the equation.

Thus, in $w + 3 = 7$, $w + 3$ and 7 are members; they are called respectively the *left* member and the *right* member, or, also, the *first* member and the *second* member.

11. Identical Equation. An equation that either involves no letters, or that is true for any values whatever that may be given to the letters involved, is called an **identical equation** or, briefly, an **identity.**

Thus, $3+4=7$, $a+b=b+a$, $5x=8x-3x$, are identities.

12. Conditional Equation. An equation that is true only on condition that the letter or letters involved have particular values is called a **conditional equation** or, briefly, an **equation.**

Thus, $w+3=7$ is an equation that is true only on condition that w represents 4.

13. The equation is the chief sentence of algebra. The identical equation is the declarative sentence and states that the two members are necessarily the same — differing, at most, in form. The conditional equation is the interrogative sentence, and asks what numbers the letters involved must represent in order that the two members may be equal.

ORAL EXERCISES

1. What must be added to 3 to make 10? State the number for which the question mark stands in $3+?=10$.

2. State the number denoted by the question mark in each case: $5+?=12$; $?+15=25$; $60=45+?$

3. State the number for which x stands in each of the following: $5+x=12$; $x+15=25$; $60=45+x$; $25+x=40$.

State the number denoted by the question mark in each case:

4. 2 times $?=12$.
5. 5 times $?=30$.
6. $\frac{1}{2}$ of $?=8$.
7. $\frac{3}{2}$ of $?=12$.
8. 2 times $?+1=9$.
9. 3 times $?+5=11$.

10. State the number represented by t in each case: $2t=8$; $4t=32$; $\frac{1}{2}t=10$; $\frac{3}{2}t=30$; $2t+1=11$.

11. A certain number less 4 is 20. What is the number?

12. A certain number less 5 is 15. What is the number?

13. x less 4 is 20. What is x? $2x$ less 5 is 25. What is $2x$?

14. In $3x+5=17$, what is $3x$? What is x?

15. In $4p-2=10$, what is $4p$? What is p?

14. Substitution. A number symbol put in place of another is said to be **substituted** for it.

For example:

$5a + 2$ becomes $5 \cdot 3 + 2$ when 3 is substituted for a; and $ax + 7$ becomes $ab + 7$ when b is substituted for x.

15. Unknown. A number symbol whose value is not known is called an **unknown number**, or simply an **unknown.**

16. Satisfying an Equation. If an equation becomes an identity when certain numbers are substituted for the unknowns, the numbers substituted are said to **satisfy** the equation.

Thus, 5 is said to *satisfy* the equation $3x = 15$, because $3 \times 5 = 15$. It is not satisfied by any other number, because 3 times any other number is not 15. Also, 7 and 5 are said to satisfy the equation $3x + 2y = 31$, because $3 \cdot 7 + 2 \cdot 5 = 31$.

17. Root of an Equation. A number that satisfies an equation is called a **root** of the equation.

18. Solving Equations. To **solve** equations is to find their roots.

ORAL EXERCISES

What number satisfies each of the following equations?

1. $3x = 6$.	7. $7y + 5 = 40$.	13. $2n = 90$.
2. $9x = 18$.	8. $2y + 1 = 3$.	14. $2y - 7 = 13$.
3. $7x = 35$.	9. $30 - 6 = 4y$.	15. $\frac{1}{8}w - 3 = 2$.
4. $4x = 32$.	10. $4w + 2 = 10$.	16. $2n = 4800$.
5. $5x + 2 = 22$.	11. $4w + 6 = 46$.	17. $2n + 1 = 27$.
6. $8x + 12 = 20$.	12. $\frac{1}{2}z + 3 = 3$.	18. $2n + 1 = 625$.

19. Preparatory.

If two weights are in balance, and if the following changes are made in one weight, what change, in each case, must be made in the other to preserve the balance?

1. Two ounces added. 2. Two ounces taken away.

3. The number of ounces in one weight made three times as great.

4. The number of ounces in one weight made $\frac{1}{4}$ as great.

20. Properties used in Solving Equations. The preceding exercises suggest the following properties:

1. *If the same number is added to equal numbers, the results are equal.*

2. *If the same number is subtracted from equal numbers, the results are equal.*

3. *If equal numbers are multiplied by the same number, the results are equal.*

4. *If equal numbers are divided by the same number (not zero), the results are equal.*

21. The following examples show how these properties are used in solving equations:

EXAMPLES

1. Solve: $3x + 5 = 23.$ *(1)*

Subtracting 5 from both members, $3x = 18.$ *(2)*

Dividing both members of *(2)* by 3, $x = 6.$ *(3)*

TEST. 6 *satisfies* $3x + 5 = 23$, because $3 \cdot 6 + 5 = 23.$

2. Solve: $p + 2 + \frac{1}{3}p = \frac{2}{3}p + 6.$ *(1)*

Subtracting $\frac{1}{3}p$ from both members, $\frac{2}{3}p + 2 = 6.$ *(2)*

Subtracting 2 from both members of *(2)*, $\frac{2}{3}p = 4.$ *(3)*

Dividing both members of *(3)* by $\frac{2}{3}$, $p = 6.$ *(4)*

TEST. 6 *satisfies (1)*, because $6 + 2 + \frac{1}{3} \cdot 6 = \frac{2}{3} \cdot 6 + 6.$
$10 = 10.$

22. Testing. The correctness of the work of solving an equation should be tested by substituting the result in the given equation. If the members become identical, the number substituted is a root of the equation.

WRITTEN EXERCISES

Solve and test:

1. $4x + 1 = 7.$
2. $3x + 1 = 10.$
3. $5x = x + 16.$
4. $2x + 7 = 27.$

5. $6y + 2 = 20.$
6. $8z + 2 = 42.$
7. $72 = 12x.$
8. $6x = 9 + 3x.$
9. $\frac{3}{2}x = 25 + .5x.$
10. $11y + 1 = 9y + 3.$
11. $3x + 2x = 4x + 16.$
12. $11m + 3 = 2m + 9 + 2m.$

23. Use of the Equation in Solving Problems. Equations may be used in solving problems. The method is made clear by examples.

EXAMPLES

1. If a certain number is doubled and 16 is added, the result is 46. What is the number?

SOLUTION. 1. Let n be the number.
2. Then $2n + 16$ is double the number plus 16.
3. But 46 is given as double the number plus 16.
4. Therefore, $2n + 16 = 46$.
5. Therefore, $2n = 30$, and $n = 15$. Why?

TEST. 15 doubled makes 30, and 30 plus 16 is 46. Thus, 15 satisfies the conditions of the problem.

2. A salesman sold twice as many articles on Friday as on Thursday, and 5 more on Saturday than on Friday; on Saturday he sold 15. How many did he sell on Thursday?

SOLUTION. 1. Let x be the number that he sold on Thursday.
2. What does $2x$ represent? $2x + 5$?
3. State two expressions, each of which is the number sold on Saturday.
4. Since $2x + 5 = 15$, $x = 5$.

TEST. $2 \cdot 5 + 5 = 15.$

Any letter may be used to represent the unknown, as y for the number of years, d for the number of dollars, r for rate, or p for the number of pounds pressure (in physics), but in algebra x is most frequently used.

24. Finding the Equation. In each solution above, step 4 contains the statement of the problem in the form of an equation. *This statement is reached by finding two expressions for the same number and using them as the members of an equation.*

25. Sign of Deduction. Instead of the word "hence," or "therefore," the sign $\therefore$ is often used. It is called the **sign of deduction.**

Thus, $\therefore 2n + 16 = 46$, is read "Therefore, $2n + 16 = 46$."

WRITTEN EXERCISES

Write the solution of each problem in steps as shown above:

1. A house and lot are worth \$4800, and the house is worth 7 times as much as the lot. Find the value of each.

2. Lucy thought of a number, doubled it, added 16, and obtained 50. Of what number did she think?

3. A tower and flagstaff are together 120 ft. high; the height of the tower is 5 times that of the flagstaff. Find the height of each.

4. $\frac{3}{5}$ of the total height of a bridge pier is out of the water, and 10 ft. of the height is under water. What is the height of the pier?

5. When goods are sold at a gain of $\frac{1}{5}$ of their cost, what is the cost of goods which sell for \$12?

6. A man's salary was increased by $\frac{1}{8}$ of itself; he then received \$1600. What was his salary before the increase?

7. A, B, and C own an automobile jointly. In a month of 30 days A used it 7 times as many days as C, B used it 4 times as many days as C, and it was unused 6 days. How many days did each use it?

8. In latitude 42° (about that of New York), the longest day (from sunrise to sunset) lacks 2 h. 48 min. of being twice as long as its night (from sunset to sunrise). Find the length of each.

SUMMARY

I. **Definitions.**

1. If two expressions represent the same number, their equality may be indicated by the sign $=$; such a statement of equality is called an *equation*. Sec. 9.

2. The *members* of an equation are the two expressions connected by the sign $=$. Sec. 10.

3. An *identical equation* is an equation that either contains no letters or is true for any values whatever that may be given to the letters involved. Sec. 11.

4. A *conditional equation* is an equation that is true only on condition that the letters involved have particular values. Sec. 12.

5. A number symbol put in place of another is said to be *substituted* for it. Sec. 14.

6. An *unknown* is a number symbol whose value is not known. Sec. 15.

7. If an equation becomes an identity when certain numbers are substituted for the unknowns, the numbers substituted are said to *satisfy* the equation. Sec. 16.

8. A *root* of an equation is a number that satisfies the equation. Sec. 17.

9. To *solve* equations is to find their roots. Sec. 18.

10. The sign $\therefore$, read "hence" or "therefore," is used to express *deduction*. Sec. 25.

II. **Properties and Processes.**

1. *Properties used in Solving Equations.* If two numbers are equal, the numbers are equal which result from: Sec. 20.

   a. Adding the same number to each.
   b. Subtracting the same number from each.
   c. Multiplying each by the same number.
   d. Dividing each by the same number (not zero).

2. The correctness of the work of solving an equation should be tested by substituting the result in the given equation. If the members become identical, the number substituted is a root of the equation. Sec. 22.

3. A problem is stated in the form of an equation by finding two expressions for the same number and using them as members of an equation. Sec. 24

4. To test the solution of a problem, substitute the result obtained by solving the problem in the conditions of the problem. If it satisfies them, the solution of the problem is correct. Sec. 23.

REVIEW

ORAL EXERCISES

Solve the equations:

1. $4x = 20$.
2. $3y + 4 = 25$.
3. $4t + 1 = 27$.
4. $2r + 5 = 13$.
5. $4v + 1 = 9$.
6. $6u = 14 - 2$.
7. $8 + 2 = 5s$.
8. $7x + 7 = 28$.
9. $14 = 6x + 2$.
10. $20 - 4 = 4y$.
11. $9r = 360$.
12. $17u = 3400$.
13. $20p = 50 - 10$.
14. $60 + 15 = 25x$.
15. $18y = 360$.

16. What is the proof that a number is the root of an equation?

17. If any process is performed on one member of an equation, how must the other member be treated in order to preserve the equality (or balance)?

WRITTEN EXERCISES

Solve and test:

1. $16t + 5 = 37$.
2. $14x = 25 + 9x$.
3. $48 = 8y + 16$.
4. $12t + 13 = 49$.
5. $3s + 2 = 19$.
6. $12t + 2 = 38$.
7. $z + 4z = 85$.
8. $15s + 2 = 12$.
9. $\frac{1}{2}x + 2 = \frac{7}{3}$.

10. The distance from New York to Chicago by a certain route is 900 miles. If a train runs this in 18 hours, what is the average speed of the train per hour? At that rate, how far would the train run in h hours? How many hours would be required to run 100 mi.? m mi.?

11. A freight train consisted of 48 cars. The number of closed cars was 6 more than twice the number of open cars. Find how many there were of each.

12. Texas exceeds California in area by 107,420 sq. mi. The sum of their areas is 424,140 sq. mi. Find the area of each.

13. The area of Kansas is twice that of Ohio. The sum of their areas is 123,000 sq. mi. Find the area of each.

CHAPTER III

FACTORS. MONOMIALS AND POLYNOMIALS

26. Product and Factors. In algebra, as in arithmetic, the result of multiplication is called a **product**, and the numbers multiplied are called the **factors** of the product.

27. Different sets of factors may have the same product.

Thus, 24 is the product of 2 and 12; or of 2, 3, and 4; or of 2, 2, and 6; or of 8, 6, and $\frac{1}{2}$, etc. $3ay$ is the product of a and $3y$, or of 3 and ay, or of $3a$ and y, etc.

28. Literal and Numerical Factors. Factors expressed by letters are called **literal** factors; factors expressed by numerals are called **numerical** factors.

For example:

$2ax$ is the product of the factors 2, a, and x.
2 is a numerical factor, a and x are literal factors.

In a product it is customary to put the numerical factor (if any) first, and the literal factors in alphabetical order.

29. In whatever order the factors are taken, the product is the same. This is called the **Commutative Law of Multiplication.**

For example, $3 \cdot a \cdot b \cdot x$ yields the same product as $ab \cdot 3 \cdot x$ and the same as $abx \cdot 3$.

30. Unity is a factor of every number, but it is not ordinarily mentioned in giving lists of factors.

Thus, a set of factors of abc are 1, a, b, and c; but 1 is usually not mentioned.

ORAL EXERCISES

State the products of the following sets of factors:

1. 3, 8.	3. a, 5.	5. c, d.	7. $\frac{1}{2}$, m, v.
2. 3, a, y.	4. 4, b.	6. v, t.	8. a, b, c.

Name a set of factors for each of the following products:

9. 10.	11. prt.	13. $\frac{1}{2}ab$.	15. gt.
10. $3a$.	12. lr.	14. mv.	16. $\frac{1}{2}sf$.

Name three sets of factors for each of the following and state which factors are literal and which are numerical:

17. 40.	19. $25ab$.	21. prt.	23. $75pq$.
18. $18a$.	20. $20hr$.	22. $25mv$.	24. $abcd$.

31. Power and Base. A product formed by using the same number one or more times as a factor is called a **power** of the repeated factor. The repeated factor is called the **base.**

32. Exponent. When a factor is to be repeated, it is usual to write the factor only once and place a small number above and to the right to show how many times the factor is to be taken. The small number is called an **exponent.**

Thus, $2 \cdot 2$ is the second power (or square) of 2, and is written 2^2; $a \cdot a \cdot a$ is the third power (or cube) of a, and is written a^3. Similarly, $3\,aabbb$ is written $3\,a^2b^3$.

33. A number without an exponent is understood to have the exponent 1, since the number is used once as a factor.

Thus, $5 = 5^1$; $a = a^1$; $7\,xyz^3 = 7^1\,x^1y^1z^3$.

34. The factors of numbers can be conveniently grouped by the use of exponents.

For example:

$$12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3. \qquad 144 = 12 \cdot 12 = 12^2.$$

$$225 = 3 \cdot 3 \cdot 5 \cdot 5 = 3^2 \cdot 5^2. \qquad 600 = 2^3 \cdot 3 \cdot 5^2.$$

35. Prime Numbers. As in arithmetic, so in algebra, an integer whose only integral factors are itself and unity is called **a prime number.**

36. Prime Factors. Prime numbers occurring as factors are called **prime factors.**

A number has only one set of *prime* factors.

ORAL EXERCISES

Name the exponents and tell what each means:

1. $2a^2x^2$. 2. $3x^3$. 3. 5^2xy^3. 4. a^4b^4. 5. $2a^5y$.

State the value of:

6. 2^3. 7. 3^2. 8. 5^2. 9. 3^3. 10. 3^4.

WRITTEN EXERCISES

Indicate the prime factors of:

1. 18. 3. 96. 5. 640. 7. 360. 9. 1225.
2. 75. 4. 128. 6. 240. 8. 275. 10. 1350.

Indicate by use of exponents the number of:

11. Things in 12 dozen.
12. Square inches in 1 sq. ft.
13. Things in a great gross.
14. Cubic inches in 1 cu. ft.
15. Days in 7 weeks.
16. Seconds in 60 minutes.
17. Years in 10 decades.
18. Years in 100 centuries.

Indicate by a power the value in cents of:

19. A cubic yard of earth at 3¢ a cubic foot.

20. A cubic foot of ore at 12¢ a cubic inch.

21. 1000 is what power of 10? 500 is 5 times what power of 10?

Using exponents, factor so that one factor is a power of 10:

22. 100. 25. 10,000. 28. 900,000.
23. 700. 26. 70,000. 29. 4,000,000.
24. 6000. 27. 810,000. 30. 1,000,000,000.

37. **Coefficient.** Any factor in a product is called the **coefficient** of the rest of the product.

Thus, in the product $3axy$, 3 is the coefficient of axy, $3a$ is the coefficient of xy, $3ax$ is the coefficient of y, $3ay$ is the coefficient of x, and the like.

38. Numerical Coefficient. A coefficient expressed in numerals is called a **numerical coefficient.**

Thus, 3 is the numerical coefficient in $3x$, and $\frac{1}{2}$ is the numerical coefficient in $\frac{1}{2}xy$.

The term "coefficient," used with no other indication, means the numerical coefficient.

39. In any product whose numerical coefficient is not expressed, the coefficient 1 is understood.

Thus, ab, abx, bc^2y, are the same as $1\,ab$, $1\,abx$, $1\,bc^2y$; and the numerical coefficient in each is 1.

ORAL EXERCISES

1. Name the coefficient of ab in $6\,ab$. The coefficient of b.

2. Name the coefficient of xy in $\frac{2}{3}axy$. Of axy. Of y.

Name the numerical coefficient in each of the following:

3.	4.	5.	6.	7.	8.	9.	10.	11.
$2x$.	$3y$.	$\frac{2}{3}ax$.	by.	$.5\,cz$.	$\frac{4}{3}my$.	$\frac{1}{2}gt^2$.	xyz.	$\frac{1}{2}mr^2$.

40. Order of Operations. In an expression containing a series of operations, multiplications and divisions are to be performed before additions and subtractions, unless otherwise indicated.

Thus, $4+5\cdot 3$ means $4+15$, or 19.

Similarly, $5+16\div 2$ means $5+8$, or 13; and $3+8\div 2-5=3+4-5=7-5=2$.

ORAL EXERCISES

Perform the operations indicated:

1. $5\cdot 4-3$.
2. $9+2\cdot 6$.
3. $14-8\div 4$.
4. $15-12\div 4$.
5. $40-34\div 2$.
6. $52\div 26-1$.
7. $10\cdot 3-3\cdot 8$.
8. $3\cdot 9-15\div 5$.
9. $5\cdot 6-18+7$.
10. $18\div 2+3\cdot 6$.
11. $34+6\div 3-16$.
12. $3+4\cdot 5-13$.
13. $a+b\div a-a$.
14. $c\cdot d+a\cdot b$.

41. The Use of the Parenthesis. In a series of operations the parenthesis may be used to indicate that certain additions and subtractions are to be performed first.

For example:

$6 + 4 \cdot 3$ means add 6 to 4 times 3, obtaining 18. But $(6 + 4) \cdot 3$ means add 6 and 4 and multiply by 3, obtaining 30. In other words, what is in the parenthesis is to be treated as a single number.

$(4 + 16) \div 2$ means $20 \div 2$, or 10, and not $4 + \frac{16}{2}$, or 12.

$8 - (12 - 7)$ means that 7 is first to be subtracted from 12 and then the result subtracted from 8. That is, $8 - (12 - 7) = 8 - 5 = 3$.

$14a - (7a + 3a) = 14a - 10a = 4a$.

42. When a number symbol is placed before or after a parenthesis with no intervening sign, multiplication is indicated.

For example:

$2a(c + 2c)$ means $2a \cdot 3c$, or $6ac$.

$(4x + 5x)4x$ means $9x \cdot 4x$ or $36x^2$.

$(a + 4a)(8b - 5b)$ means $5a \cdot 3b$ or $15ab$.

$3(b + 2c) - c$ means $3b + 6c - c$, the multiplication by 3 being performed before c is subtracted.

$5(100 - x) + 25$ means $500 - 5x + 25$, or $525 - 5x$, the multiplication by 5 being performed before 25 is added.

ORAL EXERCISES

Perform the operations indicated:

1. $(15 - 6) \div 3$.
2. $7(25 + 5)$.
3. $(18 - 12) \div 6$.
4. $5(6 + 5 - 9)$.
5. $(2 + 5) \cdot (5 - 3)$.
6. $(24 + 6) \div (8 - 3)$.
7. $(2a + a) \div 3$.
8. $(2a + 3a)b$.
9. $(2a + 3a) \div (3b + 2b)$.
10. $a - (b + 2b - 3b)$.
11. $ab(3c - 2c) + d$.
12. $m(m + 5m) - 2m$.
13. $(2a + a) \cdot (3c - c)$.
14. $x(5x - 2x) - y(y + 4y)$.
15. $19 - (4 + 7)$.
16. $8a - (7a - 3a)$.
17. $43x + (28x - 8x)$.
18. $86y - (10y - 4y) + 10y$.

19. $18-(7-3)$.
20 $46-(27+7)$.
21. $14a-(9a-3a)$.
22. $28q-(15q-5q)$.
23. $(15-7)ax$.
24. $(7+5)(8-6)$.
25. $(2a+7a)(3x+x)$.
26. $3a(9x-7x)$.
27. $(2x+3x)(3x-2x)$.
28. $(12x-7x)(12y-7y)$.

43. Symbols of Grouping. The parenthesis is used to indicate that the number symbols grouped within it are to be taken as a single number. Other symbols of grouping are $\{\ \}$, $[\ ]$, $\overline{\quad\quad}$; these have the same meaning as the parenthesis. The bar of the fraction may also be a symbol of grouping.

Thus, in $\frac{a+b}{c+d}$, the bar groups $a+b$ into one number, and $c+d$ into one number. The fraction means $(a+b)\div(c+d)$.

44. Use of the Parenthesis in Stating Equations. In solving problems by means of equations, the parenthesis is often used.

EXAMPLE

The sum of two numbers is 100, and 3 times one of them is 7 times the other; what are the numbers?

SOLUTION. 1. Let x be the smaller number.
2. Then, $100-x$ is the larger number.
3. Then, $7x$ is 7 times the smaller one and $3(100-x)$ is 3 times the larger one.
4. $\therefore 7x=3(100-x)$, according to the problem.
5. $\therefore 7x=300-3x$. (Sec. 42.)
6. $\therefore 10x=300$. $\therefore x=30$, and the numbers are 30, 70.

TEST. $30+70=100$, and $7\times 30=3\times 70$. Therefore the numbers found fulfill the conditions of the problem.

The use of the parenthesis is seen in the third and fourth steps.

WRITTEN EXERCISES

1. Write an equation stating that if the cost (c) of a lot be diminished by \$ 200 and the remainder multiplied by 5, the result will be the value (v) of the house.

2. Write the equation which states that c times the sum of x and a equals d.

3. In a certain postoffice there are three rates of pay: \$ 50, \$ 100, and \$ 150 per month. There are 5 more men receiving \$ 100 than \$ 150, and 2 more receiving \$ 50 than \$ 100; the monthly pay roll is \$ 1150. Letting x represent the number receiving \$ 150, write the equation needed to find x.

4. Two men enter a partnership and furnish a capital of \$ 5000; twice what one furnishes is 3 times what the other furnishes. How much does each furnish?

5. The sum of two numbers is 40; the smaller number, x, is $\frac{1}{3}$ of the larger number. Write the equation needed to find x. Find the numbers.

6. A rectangular lot is 20 ft. longer than it is wide. Using x to represent the width, state what represents the length. Write an equation stating that 4 times the width equals 2 times the length. Find the dimensions of the lot.

45. Monomials. A **monomial** is an algebraic expression within which no operation of addition or subtraction is indicated, unless within a symbol of grouping.

Thus, a, ab, $a \div 2b$, $\frac{5ab}{c^2}$, $7(a+b)$, $\frac{2c-5}{x}$, are monomials.

46. Polynomials. An algebraic expression consisting of two or more monomials connected by the sign + or −, is called a **polynomial.** The monomials are called the **terms** of the polynomial.

Thus, $a + 5b + c + \frac{d}{2}$ is a polynomial whose terms are a, $5b$, c, and $\frac{d}{2}$.

47. Binomials. A polynomial of two terms is called a **binomial.**

Thus, $a + b$, $b^2 - c$, $xy + m$, $3b^2 - a$, $\frac{cd}{2} - \frac{z}{w^2}$, are binomials.

48. Trinomials. A polynomial of three terms is called a **trinomial.**

Thus, $a + b + c$, $a + 2b - c$, $\frac{x}{y} - ab + 3c$, are trinomials.

49. Compound Terms. Expressions are sometimes grouped into **compound terms.**

Thus, $3a - 2b + c + d$ may be grouped into the trinomial $3a - 2b + (c+d)$, in which $(c + d)$ is a compound term.

ORAL EXERCISES

Name the monomials in the following list; the binomials; the trinomials:

1. $a+b-c$.
2. $4x^2+7$.
3. $a+b+c+d$.
4. $\frac{1}{2}gt^2$.
5. gt^2+a.
6. mv^2.
7. $a+5b-c$.
8. $x-y+z+w$.
9. $5x^2+ay^3$.
10. $2hr$.
11. prt.
12. $2g-5+x^3$.

13. What is the coefficient of t^2 in Exercise 4?

14. What is the coefficient of v^2 in Exercise 6?

15. What is the numerical coefficient in Exercise 4?

16. There are x feet of lumber in one pile and y feet in another. How many feet are there in both piles? What kind of polynomial represents this number?

17. A man has a dollars in one bank, b dollars in another, and c dollars in a third. What kind of polynomial represents his money in the three banks?

18. The base of a rectangle is b, and its altitude is twice the base. What is its perimeter? (See p. 5, Exc. 18.) What kind of algebraic expression is this?

19. A man earned d dollars per day and his son c dollars; the father worked 6 days per week and the son 2 days. How much did the two earn per week? What kind of polynomial expresses this amount?

WRITTEN EXERCISES

1. Write three monomials.

2. Write three binomials. Three trinomials.

Rewrite these polynomials, using exponents where possible:

3. $a+bb$.
4. $2aa+b$.
5. $aa-bbb$.
6. $ccc-bb$.
7. $3aayyy$.
8. $2\cdot 2\cdot 2bbb$.
9. $15mvvq$.
10. $16xxy-cd$.
11. $100aabb-sss$.

Using $a=1$, $b=2$, and $c=3$, find the value of each of the following polynomials:

12. $5a+9b$.	18. $9b+2a-c$.	24. a^2+b^2.
13. $10a-5b$.	19. $2a+3b-2c$.	25. b^3-a^2.
14. $2a+b-c$.	20. $3a+7b+11c$.	26. ac^2+3b^3.
15. $3a+15b$.	21. $3a-7b+11c$.	27. $7ab^2-c^3$.
16. $2a+3b+3c$.	22. $61b-2c-20a$.	28. c^3-b^3.
17. $.9a+.3b-.1c$.	23. $\frac{1}{2}a+\frac{3}{8}c-\frac{1}{6}b$.	29. c^4-1+5b^3.

30. A fruit grower picked $2a$ bu. of apples, $3b$ bu. of peaches, and $4c$ bu. of plums. Write the polynomial that expresses the number of bushels of fruit that he picked.

50. Uses of Monomials. Monomials have various uses. For example:

1. *They are used as formulas in business arithmetic.*

Thus:

br is often used as a short way of stating *base times rate* in percentage.

prt is often used as a short way of stating *principal times rate times time* in interest.

lr is often used as a short way of stating *list price times rate* in discount.

2. *They are used as formulas of measurement.*

Thus:

ab is often used as a short way of stating *altitude times base* in finding areas.

abc is often used as a short way of stating *length times breadth times thickness*, in finding volumes.

π (read "pi") is used to denote *the number by which the length of the diameter of a circle must be multiplied to produce the length of the circle.*

The value of π is approximately 3.1416. Letting $l=$ length of circle, and $d=$ diameter, we have $l=\pi d=3.1416\,d$.

What precedes implies that the circle is defined as a curve, rather than a surface. This definition is preferable because it is the one used in advanced mathematics, in other sciences, and in common parlance.

3. *They are used to express laws of physics.*

Thus, *vt* is often used as a short way of stating *product of velocity and time* in finding distance.

WRITTEN EXERCISES

1. Percentage $= br$. Find the percentage when $b = 400$ and $r = 30\%$.

2. Rate $= \frac{p}{b}$. Find the rate when $p = 15$ and $b = 750$.

3. Principal $= \frac{i}{rt}$. Find the principal when $i = \$500$, $r = 5\%$, and $t = 10$ yr.

4. Interest $= prt$. Find the interest when $p = \$100$, $r = 5\%$, and $t = 5$ yr.

5. Discount $= lr$. Find the discount when $l = \$820$ and $r = 12\frac{1}{2}\%$.

6. Rate of discount $= \frac{d}{l}$. Find the rate when $d = \$34$ and $l = \$875$.

7. The area of a rectangle $= ab$. Find the area when $a =$ 20 in. and $b = 17\frac{1}{2}$ in.

8. What is the area of a rectangle when $a = 4x$ inches and $b = 3x$ inches?

9. Copy the following table and fill out the blanks, using the value of π mentioned in Sec. 50. Answer from your table:

d	πd
2 in.	6.2832 in.
3 in.	( )
$\frac{1}{2}$ in.	( )
1.5 in.	( )
10 ft.	( )

(*a*) What is the length of a circle whose diameter (d) is 3 in.?

(*b*) Of one whose diameter is 1.5 in.?

(*c*) Of one whose diameter is $\frac{1}{2}$ in.? 10 ft.?

10. The distance (d) traveled by a body in time (t) moving with velocity (v) is vt. Copy the following table and fill out the blanks. Answer from your table:

v	t	$vt = d$
30	2	60
50	2	( )
36	$1\frac{1}{4}$	( )
400	5.5	( )
150	17	( )

(*a*) How far will a train moving 30 ft. per second go in 2 sec.?

(*b*) How far will a train moving 50 mi. per hour travel in 2 hr.?

(*c*) How far will a bullet traveling 400 ft. per sec. go in 5.5 sec.?

11. The number of square units in the area of a triangle is $\frac{1}{2}$ of the product of the numbers of linear units in its altitude (a) and base (b). Copy this table and fill out the blanks.

a	b	$\frac{1}{2}ab$
6	5	15
10.5	8	( )
70	9	( )
3.3	5	( )
9	1.7	( )

Answer these questions from your table:

(a) What is the area of a triangle of altitude 10.5 in. and base 8 in.?

(b) What is the area of a triangle of altitude 70 ft. and base 9 ft.?

(c) What is the area of a triangle of altitude 3.3 yd. and base 5 yd.?

51. Uses of Polynomials. Polynomials, like monomials, have various uses as formulas.

For example:

$l - lr$ may stand for *list price – discount*, or net price.

$c + rc$ may stand for *cost + rate of gain times the cost*, or the selling price.

$2a + 2b$ may stand for *the perimeter of a rectangle of sides a and b.*

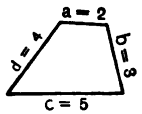
$2ab + 2ac + 2bc$ may stand for *the surface of a rectangular prism of edges a, b, c.*

$t_1 - t_2$ may stand for the *difference between the first and second readings* of a thermometer; $t_1 - t_2$ is read "t sub-one minus t sub-two," or simply "t one minus t two." (The numbers written below to distinguish between the different values of t are called **subscripts**.)

ORAL EXERCISES

1. What is the value of $l - rl$ when $l = \$100$ and $r = 5\%$? What is the net price of goods listed at \$100 and bought at a discount of 5%?

2. What is the value of $c + cr$, when $c = \$200$ and $r = 10\%$? What is the selling price of goods which cost \$200 and are sold at a gain of 10%?



3. When $a = 3$, $b = 4$, $c = 5$, $d = 6$, what is the value of $a + b + c + d$?

4. What is the value of $a + b + c + d$ in the figure?

5. What is the value of $2a + 2b$ when $a = 3$, $b = 5$? What is the perimeter of a rectangle of sides 3 yd. and 5 yd.?

WRITTEN EXERCISES

1. When $a = 10$, $b = 15$, $c = 24$, find the value of $2ab + 2ac + 2bc$. What is the area of the surface of a rectangular prism whose edges are 20 in., 15 in., 50 in.?

2. Find the value of $2ab + 2bc + 2ac$, when $a = 20$, $b = 25$, $c = 50$.

3. Find the value of $t_1 - t_2$, when $t_1 = 32$ and $t_2 = 28$.

4. Find the value of $t_1 + t_2$, when $t_1 = 40$ and $t_2 = 60$.

5. Find the value of $a^2 + b^2$, when $a = 13$, $b = 210$; also when $a = 75$, $b = 100$.

6. Find the value of $a^2 + b^2 + c^2$, when $a = 35$, $b = 20$, $c = 65$; also when $a = 100$, $b = 75$, $c = 150$.

7. Find the value of $ut + \frac{1}{2}at^2$, when $u = 1500$, $a = 200$, $t = 10$.

52. Degree of a Monomial. The degree of a monomial is the sum of the exponents of its literal factors.

Thus: a^2 is of the second degree.
$3ab$ is of the second degree.
$2a^3b$ is of the fourth degree.

But the degree is often expressed with respect to some letter or letters.

Thus, $3ax^2y^3$ is of the first degree with respect to a, of the second degree with respect to x, of the third degree with respect to y, and of the fifth degree with respect to x and y.

53. Degree of a Polynomial. The degree of a polynomial is that of its term of highest degree; its degree with respect to a letter is the highest degree of that letter in the polynomial.

Thus, $a^3x^2 - 5by + xy^4z$ is of the sixth degree; it is of the third degree in a, the first in b and in z, the second in x, the fourth in y, and the sixth in x, y and z.

NOTE. It is not necessary in elementary algebra to define the degree of expressions containing radicals or fractions.

ORAL EXERCISES

State the degree of each of the following monomials:

1. a^2b.	4. $3a^2x$.	7. $5mn^2$.	10. x^2yz^2.
2. ax.	5. a^3b.	8. $9xyz$.	11. $\frac{1}{2}mv$.
3. $2ax$.	6. $4a^2b^2$.	9. $9x^2bz$.	12. $\frac{1}{2}gt^2$.

13. State the degree of the expressions in Exercises 1–6 with respect to a; in Exercises 8–10 with respect to x.

14. State the degree of the expressions in Exercises 4–9 with respect to each letter involved.

State the degree of each polynomial; also its degree with respect to each letter:

15. $ab^2 + b$.	17. $3x^2 + 2x + 1$.	19. $3ax + 3x^2y + y^3$.
16. $a^2b + a^2c + d$.	18. $abc + c^3 + bcd$.	20. $\frac{1}{2}m^2 + n^2 + 3pq$.

SUMMARY

I. **Definitions.**

1. The *product* of two or more numbers is the result of multiplying them together. Sec. 26.

2. The *factors* of a product are the numbers multiplied to produce the product. Sec. 26.

3. A *numerical factor* is a factor indicated by numerals.

4. A *literal factor* is a factor indicated by letters. Sec. 28.

5. An *exponent* is a small number written above and to the right of a factor. It indicates that the factor is to be repeated as many times as there are units in the exponent. Sec. 32.

6. The factor whose repetition is indicated by an exponent is called the *base*. Sec. 31.

7. A product formed by using the same number one or more times as a factor is called a *power* of the repeated factor. Sec. 31.

8. A *prime number* is an integer whose only integral factors are itself and unity. Sec. 35.

9. Any factor in a product is the *coefficient* of the rest of the product. Sec. 37.

10. A *numerical coefficient* is a coefficient expressed in numerals. Sec. 38.

11. A *monomial* is an algebraic expression within which no operation of addition or subtraction is indicated unless within a symbol of grouping. Sec. 45.

12. A *polynomial* is an algebraic expression consisting of two or more monomials connected by the signs + or −. The monomials are called the terms of the polynomial. Sec. 46.

13. A *binomial* is a polynomial of two terms. Sec. 47.

14. A *trinomial* is a polynomial of three terms. Sec. 48.

15. A *compound term* consists of two or more monomials grouped together by a sign of grouping. Sec. 49.

16. The *degree of a monomial* is the sum of the exponents of its literal factors. Sec. 52.

17. The *degree with respect to a given letter* is the exponent of that letter in the monomial. Sec. 52.

18. The *degree of a polynomial* is that of its term of highest degree. Its degree with respect to a letter is the highest degree of that letter in the polynomial. Sec. 53.

19. The fact that any product remains the same, no matter in what order its factors are taken, is called the *commutative law of multiplication*. Sec. 29.

II. Notations and Processes.

1. In a product it is customary to place the numerical factor first, and to arrange the literal factors in alphabetical order. Sec. 28.

2. *Exponents* are used to indicate the repetition of factors.

3. When *no exponent* is written, the exponent 1 is understood. Sec. 33.

4. In an expression containing a series of operations, multiplications and divisions are to be performed before additions and subtractions, unless otherwise indicated. Sec. 40.

5. The *parenthesis* may be used to indicate that addition and subtraction are to be performed first, or that whatever is within the parenthesis is to be treated as a single number. Other symbols of grouping are { }, [ ], $\overline{\quad}$, and the bar of the fraction.
Secs. 41 and 43.

6. When a number symbol is placed before or after a parenthesis with no intervening sign, multiplication is indicated.
Sec. 42.

REVIEW

ORAL EXERCISES

State the product of each set of factors:

1. $8, 6, a$. 2. $b, x, 3$. 3. $a, y, 3, 4$. 4. $2, x, a, x$.

Name three sets of factors for each of the following:

5. $24\,mx$. 6. ax^2y. 7. $\frac{1}{2}gt^2$. 8. $21\,abc$.

In each of the following name (1) the coefficient of x; (2) the numerical coefficient:

9. $4\,ax$. 10. $12\,bx$. 11. acx. 12. $5\,mxy$. 13. $12\,c^2x$.

State the value of:

14. 2^4. 15. 7^2. 16. 5^3. 17. $3^2 \cdot 2^4$. 18. $2^3 \cdot 5^2$.

From the following list select by number the binomials; the trinomials; the monomials:

19. ax^2.
20. $a - x$.
21. $2\,x^3 - 5a$.
22. $6\,a + 7\,xy$.
23. $3\,ay^2 - 4\,x + 1$.
24. $2\,ab + 7x^3 - 5\,x^2$.
25. $a^2 - 2\,ab + b^2$.
26. $x^3 + 3\,x^2 + 3\,x + 1$.

27. State the degree of each expression in Exercises 19–26 with respect to x. With respect to a.

28. The sides of a triangle are $3\,a$, $2\,b$, $5\,c$. What is its perimeter? What kind of polynomial is this?

29. What is the value of $c + cr$ when $c = \$500$ and $r = 20\%$?

30. What is the value of $x^2 + 5\,a$ when $x = 3$ and $a = 2$?

State the results of the indicated operations:

31. $39 \div 13 + 1$. 35. $(8+5)2$. 39. $8x+5x-x$.

32. $8+5\cdot 2$. 36. $7(8-6)$. 40. $8-(5+2)$.

33. $7\cdot 8-6$. 37. $8-5+2$. 41. $7a-(3a+2a)$.

34. $39\div(13+1)$. 38. $7a-3a+2a$. 42. $8x+(5x-x)$.

WRITTEN EXERCISES

Indicate the prime factors of the following, using exponents where possible:

1. 88. 2. 144. 3. 200. 4. 525.

Factor so that one factor of each is a power of 10:

5. 1000. 6. 900. 7. 23,000. 8. 3,000,000.

Find the value of each of the following when $a=2$, $b=3$, $x=1$:

9. $4ax+1$. 10. $\dfrac{3ax^2-1}{2b}$. 11. $\dfrac{b^4-a^4}{x+1}$. 12. $\dfrac{a^4+4}{b+2x}$.

13. The area of a triangle is $\frac{1}{2}ab$. What is the area of a triangle in which $a=3n$ feet and $b=14n$ feet?

14. Find the value of $a^2+2ab+b^2$ when $a=7$, $b=3$.

15. Find the value of $a^3-3a^2x+3ax^2-x^3$ when $a=7$, $x=2$.

16. The sum of two numbers is 60; the smaller, x, is $\frac{3}{5}$ of the larger. Write the equation that is necessary to find x.

17. Write the equation which states that 3 times the result of subtracting d from c is c times the result of adding 3 to d.

18. Two men enter into partnership with a joint capital of $10,000. Thirteen times what one furnishes is 7 times what the other furnishes. How much does each furnish?

CHAPTER IV

RELATIVE NUMBERS

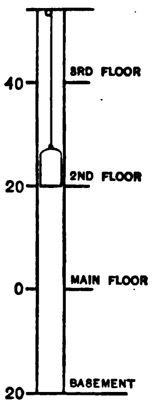
54. Relative Numbers. The preceding work is much like that of arithmetic; in fact, it might be called *literal arithmetic*. We take up now a class of numbers belonging to algebra proper, and called **relative numbers**. The following examples will illustrate them.

55. PREPARATORY.

1. *Distances upward and downward.*

The figure shows an elevator and the various floors of a building, and the distance (in feet) of the other floors from the main floor.

1. In the picture how high is the bottom of the elevator cage above the main floor? How high will it be when it gets to the third floor?



2. If the elevator cage goes from the main floor to the third floor and back to the second, how many feet is it above the main floor?

3. 40 ft. up the elevator shaft + 20 ft. down the shaft is the same as how many feet up the shaft?

4. If the elevator cage goes from the main floor to the third floor and then down to the basement, how many feet is it from the main floor? In which direction?

5. 40 ft. up the elevator shaft + 60 ft. down the shaft is the same as how many feet down?

6. 60 ft. up + 40 ft. down is the same as how many feet up?

7. 20 ft. up + 40 ft. down is the same as how many feet down?

8. 40 ft. up + 40 ft. down is the same as how many feet up?

2. *Distances to right and left.*

In a similar way state to what each of the following is equivalent:

1. 40 rd. traveled to the right + 20 rd. traveled to the left.
2. 20 rd. traveled to the right + 20 rd. traveled to the left.
3. 20 rd. traveled to the right + 50 rd. traveled to the left.
4. 30 rd. traveled to the left + 60 rd. traveled to the right.

3. *Rise and fall of temperature.*

1. On Tuesday the temperature at a certain place rose 15° above Monday's; on Wednesday it fell 10°. The two changes resulted in a temperature how much above Monday's?

2. Calling rise of temperature R and fall of temperature F, $15°R + 10°F$ is the same as (?)R.

Similarly:

3. $10°R + 15°F = ?$
4. $35°F + 45°R = ?$
5. $30°F + 15°R = ?$
6. $40°R + 40°F = ?$

4. *Amounts gained and lost.*

1. The adjoining table shows an account of gain and loss.

$\$22$ gain + $\$26$ loss = ? loss.

Gain	Loss
\$10 12	\$ 5 21
22	26
	4

Similarly:

2. $\$40\,G + \$30\,L = ?$
3. $\$17\,G + \$17\,L = ?$
4. $\$35\,L + \$20\,G = ?$
5. $\$45\,G + \$15\,L = ?$

56. Relative Numbers. In each of the preceding illustrations we have considered quantities which had two opposite directions, or **senses.** Numbers which measure quantities having opposite senses are called **relative numbers.**

It is customary in algebra to distinguish two opposite senses by calling one **positive** and the other **negative.** Either may be called positive, but the opposite to the positive is always called negative.

For example:

If distance *upward* is called *positive*, distance *downward* is called *negative*.

If a *rise* of temperature is called *positive*, a *fall* of temperature is called *negative*, and the like.

57. Positive and Negative Number. A number that measures a quantity taken in the positive sense is called a **positive number**; one that measures a quantity taken in the negative sense is called a **negative number.**

ORAL EXERCISES

What must be taken as negative when each of the following is taken as positive?

1. Degrees of latitude measured northward.
2. Number of feet to the right.
3. Number of dollars gain.
4. Number of points won.
5. Degrees of rising temperature.
6. Number of miles southward.
7. Number of people entering a car.
8. Number of pounds lifted by a balloon.

58. Notation for Positive and Negative Numbers. It is the property of relative numbers that a certain number of units taken in one sense neutralizes the same number taken in the opposite sense. This property is found in numbers to be added or subtracted. That is, a number added is offset by the same number subtracted. Such numbers are therefore relative numbers; numbers to be added are called positive, those to be subtracted are called negative, and the signs $+$ and $-$ are used to designate positive and negative numbers respectively.

Thus:

$+3$ means 3 positive units, and denotes 3 units to be added.

-3 means 3 negative units, and denotes 3 units to be subtracted.

Similarly, $+a$ means positive a, or a units to be added, and $-a$ means negative a, or a units to be subtracted.

59. Signs of Character and Signs of Operation. In algebra the signs $+$, $-$, are used to indicate the *operations* of adding or subtracting numbers, and also to indicate the positive or negative *character* of numbers.

The possibility of confusing these two uses is avoided by the following agreements:

I. *If used where a sign of operation is needed, the signs $+$, $-$, shall be regarded as signs of operation.*

For example:

In $8-5$, $-$ is a sign of operation (subtraction).

In $-8+5$, $-$ is a sign of character, because no sign of operation is needed before the 8, but $+$ is a sign of operation.

In the problem, "Add -8 and $+5$," both the signs are signs of character, because no sign of operation is needed; the operation has already been named.

II. *If it is necessary to distinguish a sign of character from a sign of operation, the former is put into a parenthesis with the number it affects.*

Thus, $-8+(-3)$ means: negative 8 plus negative 3.

III. *When no sign of character is expressed, the sign plus is understood.*

Thus, $5-3$ means: positive 5 minus positive 3.

Similarly, $8a+9a$ means: positive $8a$ plus positive $9a$.

60. Signed Numbers. In algebra every number has either the sign $+$ or the sign $-$. Consequently the numbers of algebra are often called **signed numbers.**

61. Absolute Value. The value of a signed number apart from its sign is called its **absolute, or numerical, value.**

Thus, 6 is the absolute value of both $+6$ and -6.

ORAL EXERCISES

Read the following in full, according to the agreements of Sec. 59:

1. $7-4$.	4. $14-(-6)$.	7. $6-8$.
2. $-6-8$.	5. $-14-(+6)$.	8. $6+8$
3. $-8+25$.	6. $-6-(-8)$.	9. $-6+(-8)$.

10. $2x + 3x$.
11. $2x - 3x$.
12. $-2x - 3x$.
13. $2x + (+3x)$.
14. $2x - (+3x)$.
15. $a + b$.
16. $a - b$.
17. $a + (-b)$.
18. $2x - (-3x)$.
19. $-2x - (-3x)$.
20. $-2x + (-3x)$.
21. $-2a - (-5c)$.

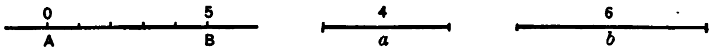
NOTE. *How to perform* the operations indicated above will be shown in the next chapter.

WRITTEN EXERCISES

Indicate, using the signs $+$, $-$:

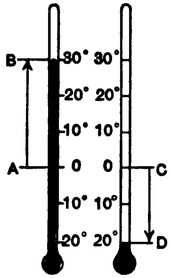
1. The sum of positive 6 and positive 4.
2. The sum of positive a and negative b.
3. The difference of positive x and positive y.
4. The difference of negative 6 and positive 3.
5. The difference of negative a and positive b.
6. The sum of negative c and negative d.

62. Number Pictures or Graphs. Numbers are often represented by lines. Such representations are called **graphs.**



Thus, the line AB represents 5, the line a represents 4, and the line b represents 6.

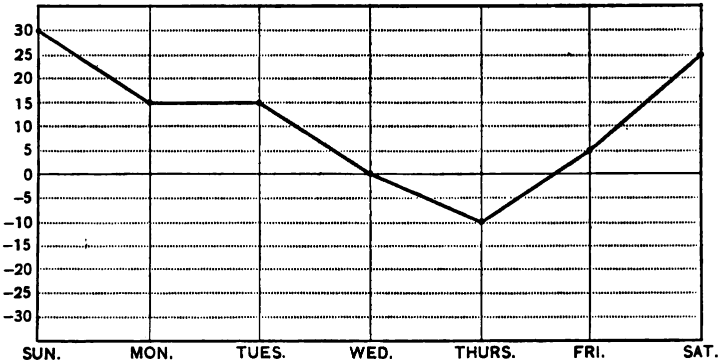
63. When relative numbers are represented by lines, the lines are drawn in opposite directions to distinguish between the positive and the negative sense.



1. On the thermometer scales here shown line AB represents 30° above zero, or $+30°$; and line CD represents 20° below zero, or $-20°$.

2. The diagram on p. 37 shows the rise and fall of the temperature for one week at a certain place. The scale represents degrees of temperature above and below zero. The diagram shows that the average temperature on Sunday was

30° above zero; that the average temperature on Tuesday was 15° above zero; and that the average temperature on



Thursday was 10° below zero. The heavy line in the figure shows the rise and fall of the average temperature from day to day.

ORAL EXERCISES

1. Using the words "plus" and "minus" for "above zero" and "below zero," respectively, read from the diagram the average temperature on Wednesday. On Saturday. On Friday.

2. On what day was the temperature highest? Lowest?

3. How much higher was the average temperature for Sunday than for Wednesday? For Monday than for Thursday? For Friday than for Thursday?

WRITTEN EXERCISES

1. The following table records the daily average temperature for the first week of January in a certain town in North Dakota. Represent the table graphically as above.

Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
10° above 0	5° above 0	0°	5° below 0	20° below 0	35° below 0	25° below 0

2. Taking the amounts above par as positive and those below par as negative, represent graphically the following prices of a stock for the consecutive days of a certain week:

MON.	TUES.	WED.	THURS.	FRI.	SAT.
5 above par	8 below par	at par	2 below par	4 below par	2 above par

Draw the broken line which indicates the variation in the price for the six days.

3. As in Exercise 2, represent graphically the following prices of a railroad stock for one week:

MON.	TUES.	WED.	THURS.	FRI.	SAT.
½ above par	1½ above par	½ below par	at par	1 above par	2½ above par

SUMMARY

Definitions.

1. *Relative numbers* measure changes in two opposite senses, so that any number of units of one sense offset, or neutralize, the same number of the opposite sense. Sec. 56.

2. The relative numbers of algebra are called *positive* and *negative numbers.* Sec. 57.

3. Positive and negative numbers mean numbers to be added and subtracted respectively. Sec. 58.

Notations.

4. The *absolute* or *numerical value* of a number is its value apart from its sign. Sec. 61.

5. Lines used to represent numbers are called *graphs.* Sec. 62.

1. In algebra the signs + and − are used to indicate the positive and negative character of numbers. Sec. 59.

2. The following rules determine whether the signs +, −, shall be regarded as signs of character or of operation:

(1) If used where a sign of operation is needed, the signs +, −, shall be regarded as signs of operation.

(2) If it is necessary to distinguish a sign of character from a sign of operation, the former is put into a parenthesis with the number it affects.

(3) When no sign of character is expressed, the sign plus is understood. Sec. 59.

REVIEW

ORAL EXERCISES

1. The temperature was $-8°$ at 6 o'clock and $+5°$ at 9 o'clock. How many degrees did it rise in this interval?

2. A ship sailed on a meridian from Lat. $+12°$ to Lat. $-2°$. In what direction did it sail and how many degrees?

Read in full:

3. $11 + 18$.
4. $14a - 9a$.
5. $-2m - (+3m)$.
6. $3y + (-2y)$.
7. $p + (-q)$.
8. $-3a - (+2b)$.
9. $xy - (-xy)$.
10. $ab - (-ab)$.
11. $mn + (-2m)$.

12. When distances measured to the right are called positive, what should the distances measured to the left be called?

WRITTEN EXERCISES

1. Write the sum of positive a and negative b.

2. Write with + and − signs: 15 dollars lost plus 10 dollars gained.

3. Represent graphically the following changes in the prices of a certain railroad stock:

Month	Jan.	Feb.	March	Apr.	May	June
Amount above par	5	3	4			2
Amount below par				1	2	

Indicate, by using the signs +, −:

4. The sum of positive x and negative y.

5. The sum of negative m and positive n.

6. The difference of positive m and negative n.

7. The difference of positive x and negative y.

CHAPTER V

ADDITION

ADDITION OF MONOMIALS

64. Like Terms. Terms or monomials that have the same literal parts, are called **like terms** or **like monomials.**

For example, the following are pairs of like terms:

ab and ab; $5a$ and $-3a$; $4ab$ and $\frac{7}{8}ab$; $2a^2b$ and $\frac{1}{3}a^2b$.

ORAL EXERCISES

From the following list select terms like the first; like the second; like the third:

1. ab^3x.	4. mp.	7. $-\frac{2}{3}ab^2$.	10. $-6\frac{1}{5}ab^2$.
2. cy^3.	5. $2b^2x^2$.	8. $5ab^3x$.	11. ax.
3. ab^2.	6. amp.	9. $-4cy^3$.	12. $24cy^2$.

65. Terms are *alike in any letter* if they contain the same power of that letter.

Thus, $2ax^2$ and bx^2 are alike in x.

Which of the terms of Exercises 1–12 are alike in a? In b? In x?

66. PREPARATORY.

1. Just as 2 ft. + 3 ft. = —— ft., so $2a + 3a =$ —— a.
2. Just as $\frac{1}{4}$ gal. + $\frac{1}{2}$ gal. = —— gal., so $\frac{1}{4}m + \frac{1}{2}m =$ —— m.
3. Just as 6 mi. + 3 mi. = —— mi., so $6y^2 + 3y^2 =$ —— y^2.
4. 4 points won + 3 points won = —— points won.
5. 4 points lost + 5 points lost = —— points lost.
6. 3° North Lat. + 8° North Lat. = —— degrees —— Lat.

7. \$5 gained + \$7 gained = —— dollars ——.

8. 8 lb. pressure + 4 lb. pressure = 12 —— ——.

9. 4 units to be added and 5 units to be added are how many units to be added?

10. 6 units to be subtracted and 3 units to be subtracted are how many units to be subtracted?

67. Addition of Numbers having Like Signs. When the numbers added are positive, the sum is positive; and when they are negative, the sum is negative.

The absolute value of the sum is the sum of the absolute values of the addends.

ORAL EXERCISES

Add:

1. $\begin{array}{r} 3a \\ \underline{6a} \\ (\ )a \end{array}$

2. $\begin{array}{r} 7b \\ \underline{3b} \end{array}$

3. $\begin{array}{r} -12x \\ \underline{-\ \ 5x} \end{array}$

4. $\begin{array}{r} 17c \\ \underline{3c} \\ (\ \ )c \end{array}$

5. $\begin{array}{r} +4 \\ \underline{+5} \end{array}$

6. $\begin{array}{r} -15y^2 \\ \underline{-10y^2} \end{array}$

7. $\begin{array}{r} +4xy \\ \underline{+7xy} \\ (\ )xy \end{array}$

8. $\begin{array}{r} -6 \\ \underline{-3} \end{array}$

9. $\begin{array}{r} -6a^2b \\ \underline{-3a^2b} \end{array}$

10. $\begin{array}{r} -4a \\ \underline{-5a} \\ (\ )a \end{array}$

11. $\begin{array}{r} +6a^2b \\ \underline{+2a^2b} \end{array}$

12. $\begin{array}{r} -7xy \\ \underline{-4xy} \end{array}$

13. $-7ab + (-3ab) = (\ )\,ab.$

14. $3abc + 5abc = (\ )\,abc.$

15. $-.7x + (-.3x) = (\ )\,x.$

16. $a^3 + 8a^3 + 5a^3 = (\ )\,a^3.$

17. $ax^2 + \frac{1}{2}ax^2 + \frac{3}{2}ax^2 = (\ )\,ax^2.$

18. $-m + (-2m) + (-3m) = (\ )\,m.$

WRITTEN EXERCISES

1. $14x + 23x + 99x = ?$

2. $40a + 75a + 89a = ?$

3. $12ab + 18ab + 75ab = ?$

4. $13xy + 50xy + 113xy = ?$

5. $49mnz + 81mnz = ?$

6. $15xy^2 + 33xy^2 + 48xy^2 = ?$

7. $4z^2 + 52z^2 + 109z^2 + 12z^2 = ?$

8. $29x^2 + 43x^2 + 87x^2 = ?$

68. Preparatory.

1. Add 5 and -3.

Regard 5 as made up of $+3$ and $+2$; then

$$\left.\begin{array}{r} +3+2 \\ \underline{-3} \\ 0+2 \end{array}\right\} \text{ or } \left\{\begin{array}{r} 5 \\ \underline{-3} \\ 2 \end{array}\right.$$

That is, the -3 offsets $+3$ of the $+5$, and the sum is $+2$.

2. Add -8 and 6.

Regard -8 as made up of -6 and -2; then

$$\left.\begin{array}{r} -6-2 \\ \underline{+6} \\ 0-2 \end{array}\right\} \text{ or } \left\{\begin{array}{r} -8 \\ \underline{6} \\ -2 \end{array}\right.$$

That is, the $+6$ offsets -6 of the -8, and the sum is -2.

69. Addition of Numbers with Unlike Signs. In adding a positive and a negative number, a positive unit and a negative unit offset each other.

The sign of the sum is that of the addend having the greater absolute value.

The absolute value of the sum is the *difference* of the absolute values of the addends.

ORAL EXERCISES

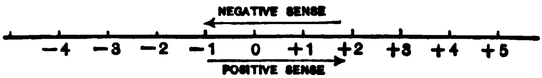
State the sums:

1. 7 negative units + 4 positive units.
2. 7 negative units + 12 positive units.
3. 8 negative units + 7 positive units.
4. 9 positive a's + 9 negative a's.
5. 10 positive x's + 15 negative x's.

6. $\begin{array}{r} -9a \\ \underline{+5a} \end{array}$	9. $\begin{array}{r} -15p \\ \underline{+10p} \end{array}$	12. $\begin{array}{r} -12mn^2 \\ \underline{+10mn^2} \end{array}$
7. $\begin{array}{r} +9b \\ \underline{-5b} \end{array}$	10. $\begin{array}{r} -3ab \\ \underline{+9ab} \end{array}$	13. $\begin{array}{r} -8x^3 \\ \underline{+9x^3} \end{array}$
8. $\begin{array}{r} +9b \\ \underline{-9b} \end{array}$	11. $\begin{array}{r} +8mn \\ \underline{-13mn} \end{array}$	14. $\begin{array}{r} +4r^2 \\ \underline{-8r^2} \end{array}$

15. $\begin{array}{r} +12\,fs \\ \underline{-16\,fs} \end{array}$ 16. $\begin{array}{r} -\frac{1}{2}\,mv^2 \\ \underline{+\quad mv^2} \end{array}$ 17. $\begin{array}{r} +.9\,y^3 \\ \underline{-.8\,y^3} \end{array}$

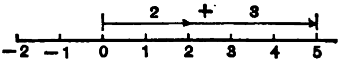
70. Graphical Addition. Positive and negative numbers may be arranged on a straight line as follows:



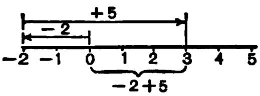
This arrangement is called the **number scale**, and it may be used to perform additions graphically.

EXAMPLES

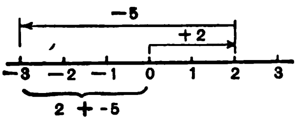
1. To add $2+3$: Let a moving point start at 0 and proceed 2 units in the positive direction (to the right), and from the place where it then is, proceed 3 units farther in the positive direction. The final position of the moving point will be distant $2+3$ units from the starting point. That is, $2+3=5$.



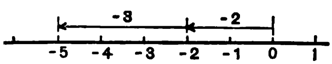
2. To add $-2+5$: Let the point proceed 2 units in the negative sense (toward the left), and from there 5 units in the positive sense. The final position is three positive units from the starting point. That is, $-2+5=3$.



3. To add $2+(-5)$: Proceed 2 units to the right, and from there 5 units to the left. The final position of the moving point is three units to the left of the starting point. That is, $2+(-5)=-3$.



4. Similarly, to add $(-2)+(-3)$ proceed two units to the left, and from there three units to the left. The final position is 5 units to the left.



That is, $(-2)+(-3)=-5$.

WRITTEN EXERCISES

Add by means of the number scale as above:

1. $3+4$.
2. $4+(-3)$.
3. $2+(-6)$.
4. $7+(-3)$.
5. $-2+5$.
6. $-2+(-5)$.
7. $-2+8$.
8. $-3+(-3)$.
9. $8+3$.
10. $-2+9$.
11. $-9+5$.
12. $-4+(-7)$.

71. Algebraic Sum. The result of adding numbers some or all of which are negative is called their **algebraic sum.**

72. The addition of several addends is similar to that of two addends. If both positive and negative terms occur, either of the following ways of adding may be used, according to convenience:

I. *Add the like terms in order.*

II. *Or, add the positive and the negative terms separately, and then combine these two sums.*

$$\begin{array}{r} 3\,a \\ -5\,a \\ 12\,a \\ \underline{-6\,a} \\ 4\,a \end{array}$$

Thus, in the column on the left, adding upward, the partial sums would be $6\,a$, a, and finally $4\,a$, the result. Or, the sum of the positive terms is $15\,a$, and that of the negative terms is $-11\,a$, and the sum of these two is $4\,a$.

73. *To add like monomials add the coefficients for the coefficient of the sum and prefix the proper sign.* Sec. 69.

WRITTEN EXERCISES

Add:

1. $\begin{array}{r} 7\,b \\ -3\,b \\ \underline{20\,b} \end{array}$ 3. $\begin{array}{r} 12\,y \\ -\ 9\,y \\ \underline{5\,y} \end{array}$ 5. $\begin{array}{r} 4\,m^3 \\ -m^3 \\ \underline{25\,m^3} \end{array}$ 7. $\begin{array}{r} -2\,x \\ 8\,x \\ \underline{-9\,x} \end{array}$ 9. $\begin{array}{r} 8\,xy^2 \\ 10\,xy^2 \\ \underline{-20\,xy^2} \end{array}$

2. $\begin{array}{r} 4\,s \\ -12\,s \\ 6\,s \\ \underline{-9\,s} \end{array}$ 4. $\begin{array}{r} -x^2y \\ 9\,x^2y \\ -2\,x^2y \\ \underline{4\,x^2y} \end{array}$ 6. $\begin{array}{r} 7\,b \\ 8\,b \\ -20\,b \\ \underline{3\,b} \end{array}$ 8. $\begin{array}{r} t^2 \\ 2\,t^2 \\ -5\,t^2 \\ \underline{25\,t^2} \end{array}$ 10. $\begin{array}{r} 20\,w \\ 30\,w \\ -19\,w \\ \underline{11\,w} \end{array}$

Solve for x:

11. $3\,x - 8\,x + 15\,x = 20.$
12. $6\,x + 8\,x - 3\,x - 4\,x - x = 21.$
13. $17\,x - 5\,x + 3\,x + 2 = 11\,x + 4\,x - 10\,x + 27.$
14. $22\,x - 3\,x - 6\,x - 4\,x = 5\,x + x + 2\,x + 9.$
15. $2\,x + 3\,a + 15\,x - 12\,x + 6\,a = 24\,a.$
16. $5\,f + 4\,x - 2\,c + 3\,x + 7\,c = 12\,x + 6\,b + 8\,c - 9\,x - 3\,c.$

ADDITION OF POLYNOMIALS

74. The addition of polynomials is similar to that of denominate numbers.

For example :

DENOMINATE NUMBERS		POLYNOMIALS	
Just as:	3 bu. 4 qt.	so:	$3b+4q$
plus	5 bu. 3 qt.	plus	$5b+3q$
equals	8 bu. 7 qt.	equals	$8b+7q$

ORAL EXERCISES

1. Add: 4 mi. 3 rd. 2 ft. also, $4m+3r+2f$
   6 mi. 7 rd. 8 ft. $6m+7r+8f$

State the numbers to fill the blanks in the following additions:

2.
$$\begin{array}{r} 2a+4b \\ 3a+5b \\ \hline (\ )a+(\ )b \end{array}$$

3.
$$\begin{array}{r} 2a+b+c \\ 5a+3b+2c \\ \hline (\ )a+(\ )b+(\ )c \end{array}$$

75. To Add Polynomials: *Arrange the like terms in columns and add as in the case of monomials, using the signs obtained as the signs of the result.*

For example :

NOT ARRANGED
$$\begin{array}{r} a+c+b \\ -3b+a+c \\ \hline \end{array}$$

ARRANGED
$$\begin{array}{r} a+b+c \\ a-3b+c \\ \hline \end{array}$$

Here the first column is $+a+a=+2a$; the second column is $+b-3b=-2b$; the third column is $+c+c=+2c$. The terms thus obtained with their signs, namely, $2a-2b+2c$, constitute the sum of the polynomials.

76. The terms of polynomials may be rearranged before adding because the sum of two or more terms is the same in whatever order the terms are taken. This is called the **Commutative Law of Addition.**

WRITTEN EXERCISES

Add:

1. $\begin{array}{r} 5a-3b \\ \underline{-\ b} \end{array}$

2. $\begin{array}{r} 2a-6c \\ \underline{3c+2a} \end{array}$

3. $\begin{array}{r} \frac{1}{2}x-\frac{1}{3}y \\ \underline{y+\frac{2}{3}x} \end{array}$

4. $\begin{array}{r} 3m-1.1n \\ \underline{6m-\ .9n} \end{array}$

5. $\begin{array}{r} \frac{2}{3}p+1\frac{4}{9}q \\ \underline{\frac{1}{3}q+\ \frac{5}{9}p} \end{array}$

6. $\begin{array}{r} -7x+\ 4y \\ \underline{9x-10y} \end{array}$

7. $\begin{array}{r} x^2+2y^2-\ z^2 \\ \underline{z^2-3y^2+2x^2} \end{array}$

8. $\begin{array}{r} at^2-6at+\ c \\ \underline{2at-8at^2+2c} \end{array}$

9. $\begin{array}{r} m^2+\ m+1 \\ \underline{m-2m^2-8} \end{array}$

10. $\begin{array}{r} p^2+\ p+8 \\ \underline{p^2+9p+6} \end{array}$

11. $\begin{array}{r} x^3+6x^2-\ 9x \\ \underline{2x^3+8x^2-10x} \end{array}$

12. $\begin{array}{r} 17p^2-\ pq+q^2 \\ \underline{-6p^2+2pq-q^2} \end{array}$

13. $\begin{array}{r} 2a+7b+11c \\ \underline{2b+\ 9c} \end{array}$

14. $\begin{array}{r} 12x+\ 8y+17z \\ \underline{9x+12z+13y} \end{array}$

15. $\begin{array}{r} \frac{1}{3}p+\ \frac{2}{3}q+\frac{3}{7}r \\ \underline{7\frac{1}{3}p+9\frac{1}{3}q+\ r} \end{array}$

77. PREPARATORY.

Find the value of each expression when each letter $=1$:

1. $a+2b$.
2. $a+b+c$.
3. $c+4d-2a$.
4. $2a-2b$.
5. $b+2c-a$.
6. $2a+3b+3c$.
7. $3a-c$.
8. $a+d+3c$.
9. $3b+c+a$.

78. Test of Addition. To test the work of addition, substitute unity for the letters. The value of the sum must equal the sum of the values of the addends.

In practice the work and test are written as follows:

SOLUTION	TEST
$2a-5b$	-3
$\underline{4a+4b}$	$\underline{+8}$
$6a-\ b$	$+5$

WRITTEN EXERCISES

Add and test:

1. $\begin{array}{r} a+\ 3b \\ \underline{11a+10b} \end{array}$

2. $\begin{array}{r} 4x\quad +z \\ \underline{2z+y+x} \end{array}$

3. $\begin{array}{r} 40m+\ n \\ \underline{5m-39n} \end{array}$

4. $\begin{array}{r} 17a+4b \\ 3a-16b \\ \hline \end{array}$

5. $\begin{array}{r} 6a+9b \\ 4a-b \\ \hline \end{array}$

6. $\begin{array}{l} 6c+d \\ 3c+2d+e \\ \hline \end{array}$

7. $\begin{array}{r} 12a+5b \\ 6a-3b \\ \hline \end{array}$

8. $\begin{array}{r} -15x+12 \\ -8x-3 \\ \hline \end{array}$

9. $\begin{array}{r} 5x^2+3x \\ -3x+7x^2 \\ \hline \end{array}$

10. $\begin{array}{r} 4xy-2z^2 \\ 5xy+10z^2 \\ \hline \end{array}$

11. $\begin{array}{r} 12t-6t^2 \\ 8t+12t^2 \\ \hline \end{array}$

12. $\begin{array}{r} 40-\frac{1}{2}gt^2 \\ 50+\frac{3}{2}gt^2 \\ \hline \end{array}$

13. $\begin{array}{r} 45m-2n+q \\ 5m+3n+q \\ \hline \end{array}$

14. $\begin{array}{r} 3a^2-5a+1 \\ 4a+8a^2-3 \\ \hline \end{array}$

15. $\begin{array}{r} 5x+y-z \\ 3x-7y+8z \\ \hline \end{array}$

16. $\begin{array}{r} \frac{1}{2}x+\frac{1}{3}y+.9z \\ \frac{1}{2}x+1\frac{2}{3}y+1.1z \\ \hline \end{array}$

17. $\begin{array}{r} 1.1a-8.9b+c \\ 3.9a+.5b-5c \\ \hline \end{array}$

18. $\begin{array}{r} x^2+xy+3y^2 \\ 6x^2+10xy+5y^2 \\ \hline \end{array}$

19. $\begin{array}{r} x^3+x^2y+8y^3 \\ 4x^2y+y^3+z^3 \\ \hline \end{array}$

20. $\begin{array}{r} 5a-7b \\ 3a+10b \\ -6a+18b \\ -7a+12b \\ \hline \end{array}$

21. $\begin{array}{r} x^2-5x \\ -4x^2+3x \\ -12x^2+x \\ 15x^2-12x \\ -10x^2+16x \\ \hline \end{array}$

22. $\begin{array}{rrr} a+ & 2b+ & 3c^2 \\ 2a & & +66c^2 \\ & 9b+ & c^2 \\ a+ & b+ & c^2 \\ 5a- & b & \\ \hline \end{array}$

23. $\begin{array}{rrrr} & p+ & 3q & \\ m+ & 3p+ & q+ & r \\ 5m+ & 2p & & +6r \\ & & 10q & +5r \\ \hline \end{array}$

24. $\begin{array}{l} a^2+4-a \\ 5-3a^2+2a \\ 6a^2+3a-5 \\ 4a-7a^2-2 \\ 8a-12-15a^2 \\ \hline \end{array}$

25. $\begin{array}{l} 4g+3v-7x \\ 5x+2y-4v \\ 2y-8g-7v \\ 13v-11x+2g \\ 12x-15g+4y \\ \hline \end{array}$

26. A dealer bought at one time 3 kinds of coal, $50a$ tons of the first kind, $10b$ tons of the second kind, and $12c$ tons of the third; at another time he bought $75a$ tons, $15b$ tons, $10c$ tons respectively of the same kinds. How many tons did he buy in all?

27. A grain dealer had in one place $8m$ bu. of oats, $5w$ bu. of wheat, and $7p$ bu. of rye; in another place $12m$ bu. of oats, $56w$ bu. of wheat. How many bushels of grain had he in both places?

28. A contractor used three kinds of lumber in a building: $30x$ feet, $10y$ feet, $16z$ feet respectively; in another building he used of the same three kinds: $50x$ feet, $39y$ feet, z feet. How many feet did he use in all?

79. The Greater of Two Numbers. Of two given numbers that one is the greater which can be produced by adding a positive number to the other. The other number is called the less.

For example:

11 is greater than 8 because it is necessary to add $+3$ to 8 to make 11.

7 is greater than -2 because it is necessary to add $+9$ to -2 to make 7.

-4 is greater than -9 because it is necessary to add $+5$ to -9 to make -4.

80. The symbol $>$ is read "is greater than," and $<$ is read "is less than."

For example:

$8 > 2$ is read "8 is greater than 2."

$-1 > -5$ is read "-1 is greater than -5."

$-5 < -1$ is read "-5 is less than -1."

ORAL EXERCISES

Read the following and state why each is correct:

1. $7 > 5$.	3. $-2 > -5$.	5. $3 < 5$.	7. $-1 < 0$.
2. $4 > -8$.	4. $0 > -7$.	6. $-4 < 2$.	8. $-8 < -6$.

WRITTEN EXERCISES

Determine which is the greater in each of the following pairs of numbers, and write the relation by use of the sign $>$:

1. 8, 6.	3. -5, 6.	5. -6, -5.	7. 0, 10.
2. 3, 4.	4. -6, 5.	6. 6, -9.	8. 0, -10.

81. Numerical Value. The greater number as above defined does not always have the greater numerical or absolute value (Sec. 61). When it is desired to speak of the numerical values only, the expression "numerically greater" is used. In distinction, the greater number is sometimes said to be "algebraically greater."

For example:

4 is greater than -9, but 4 is numerically less than -9.

-6 is greater than -15, but -15 is numerically greater than -6.

-2 is algebraically greater than -12, but numerically less than it.

ORAL EXERCISES

From the following list select the numbers that are:

a. Greater than 6.

b. Less than -5.

c. Greater than -4.

d. Numerically greater than -4.

e. Numerically greater than 6.

f. Numerically less than -5.

1. 7.	5. -18.	9. -3.	13. 5.
2. -10.	6. -1.	10. 2.	14. -6.
3. -8.	7. 0.	11. -4.	15. $-\frac{1}{2}$.
4. 12.	8. 1.	12. 3.	16. -11.

17. State the absolute value of each of the numbers in Exercises 1–16.

SUMMARY

I. Definitions.

1. *Like terms* or *like monomials* are those which have the same literal parts. Sec. 64.

2. Terms are *alike in any letter*, if they contain the same power of the letter. Sec. 65.

3. The result of adding numbers of which some or all are negative is called their *algebraic sum*. Sec. 71.

4. Of two given numbers, that one is the *greater* which can be produced by adding a positive number to the other. The other number is called the less. Sec. 79.

5. The symbol $>$ is read "is greater than," and $<$ is read "is less than." Sec. 80.

6. Of two numbers, that one is *numerically greater* which has the larger absolute value. Sec. 81.

7. The terms of polynomials may be rearranged before adding, because the sum of two or more terms is the same in whatever order the terms are taken. This is called the *Commutative Law of Addition.* Sec. 76.

II. **Processes.**

1. When the numbers added are positive the sum is positive; and when they are negative the sum is negative.

The absolute value of the sum is the sum of the absolute values of the addends. Sec. 67.

2. In adding a positive and a negative number, a positive unit and a negative unit offset each other.

The sign of the sum is that of the addend having the greater absolute value.

The absolute value of the sum is the *difference* of the absolute values of the addends. Sec. 69.

3. The addition of several addends is similar to that of two addends. If both positive and negative terms occur, either of the following ways of adding may be used, according to convenience:

(1) Add the like terms in order.

(2) Add the positive and the negative terms separately, and then combine these two sums. Sec. 72.

4. To add polynomials, arrange the like terms in columns and add as in case of monomials, using the signs obtained as the signs of the result. Sec. 75.

5. To test addition use arbitrary values. The sum of the values of the addends should equal the value of their sum. Sec. 78.

REVIEW

ORAL EXERCISES

Add:

1. $$\begin{array}{r} -6\,ab \\ -8\,ab \\ \hline \end{array}$$

2. $$\begin{array}{r} 12\,ac^2 \\ -\ \ 8\,ac^2 \\ \hline \end{array}$$

3. $$\begin{array}{r} mr^2 \\ -mr^2 \\ \hline \end{array}$$

4. $$\begin{array}{r} \frac{1}{2}\,mv^2 \\ -\ \ mv^2 \\ \hline \end{array}$$

5. $$\begin{array}{r} -xy \\ 10\,xy \\ -15\,xy \\ 11\,xy \\ \hline \end{array}$$

6. $$\begin{array}{r} 4\,\pi r^2 \\ 2\,\pi r^2 \\ \hline \end{array}$$

7. $$\begin{array}{r} 12\,xyz \\ 13\,xyz \\ \hline \end{array}$$

8. $$\begin{array}{r} -\ \ mpq \\ -6\,mpq \\ \hline \end{array}$$

9. $$\begin{array}{r} -\frac{1}{2}\,mv^2 \\ -2\,mv^2 \\ \hline \end{array}$$

10. $$\begin{array}{r} 43\,w^3 \\ -23\,w^3 \\ 20\,w^3 \\ -10\,w^3 \\ \hline \end{array}$$

11. $$\begin{array}{r} -9\,pq^2 \\ -7\,pq^2 \\ \hline \end{array}$$

12. $$\begin{array}{r} \frac{1}{8}\,\pi r^3 \\ \frac{5}{8}\,\pi r^3 \\ \hline \end{array}$$

13. $$\begin{array}{r} -6\,a^3b \\ 8\,a^3b \\ \hline \end{array}$$

14. $$\begin{array}{r} 4\,xy^2 \\ -7\,xy^2 \\ \hline \end{array}$$

15. $$\begin{array}{r} 16\,uv^2 \\ -9\,uv^2 \\ -7\,uv^2 \\ 25\,uv^2 \\ \hline \end{array}$$

16. $$\begin{array}{l} 7\,x^2-2\,x-5 \\ 4\,x^2+4\,x+5 \\ \hline \end{array}$$

17. $$\begin{array}{l} 4.5\,m+3.2\,n+\ \ p \\ .5\,m \qquad\qquad +.1\,p \\ \hline \end{array}$$

18. $$\begin{array}{l} a^3+3\,a^2b+3\,ab^2+b^3 \\ a^3-3\,a^2b+3\,ab^2-b^3 \\ \hline \end{array}$$

19. $$\begin{array}{l} 3\,x-4\,y+7\,z-9 \\ 8\,y-2\,z+4\,x-3 \\ \hline \end{array}$$

20. $$\begin{array}{rrr} x^2- & 5\,x+ & 6 \\ 2\,x^2- & 9\,x- & 1 \\ 8\,x^2+ & 12\,x & \\ 5\,x^2 & & -20 \\ 14\,x^2- & 63\,x+ & 17 \\ \hline \end{array}$$

21. $$\begin{array}{rrr} t^2+ & 3\,t- & 8 \\ -\ \ 7\,t^2+ & 11\,t+ & 3 \\ -14\,t^2- & 27\,t- & 1 \\ -23\,t^2+ & t- & 15 \\ 17\,t\ - & 23\,t- & 32 \\ \hline \end{array}$$

22. A fruit grower gathered $4\,a$ bu. of apples, $2\,b$ bu. of pears, and $2\,c$ bu. of peaches in one season. How many bushels did he gather?

23. The same man gathered $4\,a$ bu. of apples, $2\,b$ bu. of pears, and c bu. of peaches the next season. How many bushels of fruit was this? How many bushels of fruit did he gather in the two seasons?

24. A workman earned $2a$ dollars in one week, $3b$ dollars the second, and c dollars the third; during the next three he earned $3a$ dollars, $5b$ dollars, and $2c$ dollars. How much did he earn during the first three weeks? During the second three weeks? How much did he earn in all?

WRITTEN EXERCISES

Add and test:

1. $a+b+c,\ 2a+b+3c,\ a+b,\ 6b+5c.$
2. $10a+9b+c,\ 9a+10c,\ \frac{1}{3}a+\frac{1}{4}b,\ 2b+c.$
3. $.9x+.3y+z,\ .1x+.7y,\ 5y+\frac{1}{2}z,\ 4x+z.$
4. $3a-2b,\ 4a+7b,\ -6a-b,\ 14a-21b.$
5. $m^2+5mp,\ 7m^2-8mp,\ -12m^2+3p^2,\ 6m^2-4mp+7p^2.$
6. $5ax+b,\ 2ax-3b,\ -8ax+7b,\ -20ax-18b.$
7. $\frac{1}{2}x-\frac{1}{3}y,\ -\frac{3}{5}x+\frac{1}{4}y,\ .7x+\frac{2}{3}y,\ -.3x+\frac{5}{6}y.$
8. $7+2x^2,\ 3x^2-1,\ 4-5x^2.$
9. $4a+7b,\ 2a-6c,\ 3c+5b,\ 4b-7a.$
10. $x^2+7x-4,\ 3x^2-5x,\ 4x^2-11x+2.$
11. $5t-3+7t^2,\ t-3t^3,\ t^3+9t^2-15+8t.$
12. $3ab+7ac,\ 5ac-2bc,\ 6bc+9ab,\ 8bc-18ac.$
13. $x+.4,\ x^2-.5,\ 3x-.7,\ x^2-.9+4x.$
14. $2y^2-4y+y^4-1,\ 8y-y^3+3y^2-15,\ 3y-7+11y^4-15y^2,\ 4y^3+12y^2-6+y^4,\ 11-y^4+y^3-8y^2.$

15. A grocer had $7a$ dollars on hand; his ten salesmen took in $4a$, $5c$, $2b$, $6a$, $3a$, $7c$, $4b$, $2c$, $5c$, $11b$, dollars respectively. How much had he then?

16. A merchant made the following bank deposits: On Monday $3a$ dollars in gold, $4b$ dollars in silver, and $9c$ dollars in notes; on Tuesday a dollars in gold and $15c$ dollars in notes; on Wednesday b dollars in silver and $12c$ dollars in notes. How much did he deposit altogether?

CHAPTER VI

SUBTRACTION

SUBTRACTION OF MONOMIALS

82. PREPARATORY.

Read and supply the blanks:

1. Just as 4 qt. − 2 qt. = —— qt., so $4a - 2a =$ —— a.
2. Just as 8 bu. − 5 bu. = —— bu., so $8b - 5b =$ —— b.
3. Just as $\frac{3}{4}$ lb. − $\frac{1}{4}$ lb. = —— lb., so $\frac{3}{4}d - \frac{1}{4}d =$ —— d.
4. $10c - 5c =$ —— c.
5. $6a - 3\frac{1}{2}a =$ —— a.
6. $6y^2 - 5y^2 =$ —— y^2.
7. $10xy - 7xy =$ —— xy.
8. $12abc - 9abc =$ —— abc.
9. $.9axy - .3axy =$ —— axy.

83. Subtraction is the process of finding the difference between two numbers, called the *minuend* and the *subtrahend*.

84. Difference. The **difference** is the number which added to the subtrahend makes the minuend.

Thus, the difference between $12bc$ and $5bc$ is $7bc$, because $5bc + 7bc = 12bc$.

ORAL EXERCISES

State the differences of the following:

1. $\begin{array}{r} 20a \\ \underline{15a} \end{array}$	4. $\begin{array}{r} 21x^2 \\ \underline{14x^2} \end{array}$	7. $\begin{array}{r} 90pq \\ \underline{45pq} \end{array}$	10. $\begin{array}{r} 40mn \\ \underline{39mn} \end{array}$
2. $\begin{array}{r} 47b \\ \underline{27b} \end{array}$	5. $\begin{array}{r} 17bc \\ \underline{9bc} \end{array}$	8. $\begin{array}{r} 50zw \\ \underline{25zw} \end{array}$	11. $\begin{array}{r} 54d \\ \underline{24d} \end{array}$
3. $\begin{array}{r} 12b^2c \\ \underline{7b^2c} \end{array}$	6. $\begin{array}{r} 39c^2 \\ \underline{19c^2} \end{array}$	9. $\begin{array}{r} 30xy \\ \underline{17xy} \end{array}$	12. $\begin{array}{r} 19abx \\ \underline{12abx} \end{array}$

85. Subtracting Negative Numbers. To subtract a negative number, we find the number which added to it makes the minuend.

Thus:	6 less $-4 = 10$	because $-4 + 10 = 6$.
	-15 less $-7 = -8$	because -7 plus $-8 = -15$.
	$4a$ less $-2a = 6a$	because $-2a + 6a = 4a$.

86. Since subtraction is the reverse of addition, we can subtract a number by adding its opposite.

For example, "to subtract 3" and "to add -3" mean the same thing. Likewise "to subtract $-5a$" and "to add $5a$" mean the same thing. Consequently algebraic subtraction may be regarded as a special case of addition.

To subtract one number from another, change the sign of the subtrahend and add the result to the minuend.

Thus, to subtract $\begin{array}{r} 30a \\ -10a \\ \hline \end{array}$ change to $\begin{array}{r} 30a \\ 10a \\ \hline \end{array}$ and add.

The pupil should as soon as possible accustom himself to make the change of sign mentally.

ORAL EXERCISES

Find the differences:

1. $\begin{array}{r} 18 \\ -99 \\ \hline \end{array}$	6. $\begin{array}{r} -12m \\ 8m \\ \hline \end{array}$	11. $\begin{array}{r} 5a \\ 8a \\ \hline \end{array}$	16. $\begin{array}{r} -14 \\ -6 \\ \hline \end{array}$
2. $\begin{array}{r} 6ab \\ 6ab \\ \hline \end{array}$	7. $\begin{array}{r} 12pq \\ -2pq \\ \hline \end{array}$	12. $\begin{array}{r} 4x \\ 7x \\ \hline \end{array}$	17. $\begin{array}{r} 16b \\ 11b \\ \hline \end{array}$
3. $\begin{array}{r} 5x \\ -15x \\ \hline \end{array}$	8. $\begin{array}{r} 6mv^2 \\ -19mv^2 \\ \hline \end{array}$	13. $\begin{array}{r} 12y \\ 3y \\ \hline \end{array}$	18. $\begin{array}{r} -14d \\ -7d \\ \hline \end{array}$
4. $\begin{array}{r} 5xy \\ -10xy \\ \hline \end{array}$	9. $\begin{array}{r} -1.5s \\ -3.5s \\ \hline \end{array}$	14. $\begin{array}{r} -8x \\ 2x \\ \hline \end{array}$	19. $\begin{array}{r} 8g \\ -3g \\ \hline \end{array}$
5. $\begin{array}{r} -3abc \\ 3abc \\ \hline \end{array}$	10. $\begin{array}{r} -4\frac{1}{2}w \\ -9\frac{1}{2}w \\ \hline \end{array}$	15. $\begin{array}{r} -8x \\ -2x \\ \hline \end{array}$	20. $\begin{array}{r} -4p \\ -7p \\ \hline \end{array}$

21. $\begin{array}{r} 2m^2 \\ \underline{-2m^2} \end{array}$

22. $\begin{array}{r} 2m^2 \\ \underline{2m^2} \end{array}$

23. $\begin{array}{r} 23 \\ \underline{-6} \end{array}$

24. $\begin{array}{r} 23 \\ \underline{6} \end{array}$

25. $\begin{array}{r} -23 \\ \underline{6} \end{array}$

26. $\begin{array}{r} -23 \\ \underline{-6} \end{array}$

27. $\begin{array}{r} 23x^3 \\ \underline{-6x^3} \end{array}$

28. $\begin{array}{r} -t \\ \underline{-3t} \end{array}$

29. $46-52$.
30. $4a-7a$.
31. $8x^2-10x^2$.
32. $8ab-15ab$.
33. $40m-46m$.
34. $13pq-15pq$.
35. $\frac{1}{2}mv^2-mv^2$.
36. $\pi r^2-4\pi r^2$.
37. $\frac{4}{3}\pi r^3-2\pi r^3$.
38. $\frac{1}{2}gt^2-gt^2$.
39. $10fs-15fs$.
40. $45rt-50rt$.

SUBTRACTION OF POLYNOMIALS

87. **The subtraction of polynomials** is similar to the subtraction of denominate numbers.

For example:

Denominate Numbers	Polynomials
Just as: 5 lb. 4 oz.	so: $5l+4z$
minus 3 lb. 3 oz.	minus $3l+3z$
equals 2 lb. 1 oz.	equals $2l+1z$

ORAL EXERCISES

Subtract:

1. $\begin{array}{r} 12 \text{ bu. } 3 \text{ pk. } 7 \text{ qt.} \\ \underline{9 \text{ bu. } 2 \text{ pk. } 4 \text{ qt.}} \end{array}$

2. $\begin{array}{r} 12b+3p+7q \\ \underline{9b+2p+4q} \end{array}$

State the numbers to fill the blanks in the following:

3. $\begin{array}{r} 6xy+8y^2 \\ \underline{3xy+2y^2} \\ (\ )xy+(\ )y^2 \end{array}$

4. $\begin{array}{r} 5a+3b^2+8c \\ \underline{a+\ b^2+5c} \\ (\ )+(\ )+(\ ) \end{array}$

Subtract:

5. $\begin{array}{r} 6a+5b \\ \underline{a+3b} \end{array}$

6. $\begin{array}{r} 16m+2n \\ \underline{12m+\ n} \end{array}$

7. $\begin{array}{r} 17c+9d \\ \underline{17c+\ d} \end{array}$

8. $\begin{array}{l} 20x+5y+z \\ \underline{9x+5y\qquad} \end{array}$

9. $\begin{array}{r} 45a+3b+12c \\ \underline{5a+\ b+\ 2c} \end{array}$

10. $\begin{array}{r} 10x^2+6y^2+5z^2 \\ \underline{5x^2+\ y^2+5z^2} \end{array}$

WRITTEN EXERCISES

Arrange the like terms in columns and subtract:

1. $\begin{array}{r} 8x + 12q + 6a \\ \underline{4a + 5x + 3q} \end{array}$

2. $\begin{array}{r} p + 3n + 45m \\ \underline{5m + 2n + p} \end{array}$

3. $\begin{array}{r} 5c + 6b + 10a \\ \underline{b + 5a + 5c} \end{array}$

4. $\begin{array}{r} 10ax + \phantom{\frac{1}{2}}mp + 5pq \\ \underline{2pq + \frac{1}{2}mp + 5ax} \end{array}$

88. To subtract a polynomial *arrange its terms under the like terms of the minuend, subtract each term from the one above it, and use the signs obtained as the signs of the result.*

$$\begin{array}{r} 2a - 2b + c \\ \underline{a + 0 - 2c} \\ a - 2b + 3c \end{array}$$

In the example, the first column is $2a - a = a$; the second column is $-2b - (+0) = -2b$; the third column is $+c - (-2c) = +3c$. The terms thus obtained with their signs constitute $a - 2b + 3c$, the difference of the polynomials. When either polynomial lacks a term to correspond to a term of the other polynomial, supply zero in its place.

89. Test of Subtraction. Use arbitrary values to test subtraction. The sum of the values of the difference and the subtrahend must equal the value of the minuend.

Solution	Test: Let each letter = 1
$3x - 4y + c$	$3 - 4 + 1 = 0$
$x + y - 3c$	$1 + 1 - 3 = -1$
$2x - 5y + 4c$	$2 - 5 + 4 = +1$

In practice it is sufficient to write the following:

Solution	Test
$3x - 4y + c$	0
$x + y - 3c$	-1
$2x - 5y + 4c$	1

WRITTEN EXERCISES

Subtract and test:

1. $\begin{array}{r} 3a + b \\ \underline{a - b} \end{array}$

2. $\begin{array}{r} 6a - 3b \\ \underline{5a + 7b} \end{array}$

3. $\begin{array}{r} \frac{1}{2}a - \frac{1}{3}b \\ \underline{a + \frac{2}{3}b} \end{array}$

4. $\begin{array}{r} 3m - .1n \\ \underline{6m - .9n} \end{array}$

5. $\begin{array}{r} \frac{2}{3}p + \frac{1}{9}q \\ \underline{\frac{2}{3}p - \frac{8}{9}q} \end{array}$

6. $\begin{array}{r} -7x + 4y \\ \underline{10x - 9y} \end{array}$

7. $\begin{array}{l} 3xy - \quad z^2 \\ \underline{5xy + 10z^2} \end{array}$	11. $\begin{array}{l} 2a + c - 2b \\ \underline{\quad a + b - 2c} \end{array}$	15. $\begin{array}{l} 3x^2 + 2x^2y^2 + z^2 \\ \underline{\quad x^2 + 2x^2y^2 \quad} \end{array}$
8. $\begin{array}{l} 12t - \quad 6t^2 \\ \underline{\ 9t - 12t^2} \end{array}$	12. $\begin{array}{l} 7x^2 - 2x + 4 \\ \underline{2x^2 + 3x - 1} \end{array}$	16. $\begin{array}{l} 4x - 3y + 8 \\ \underline{2y + 5z - 1} \end{array}$
9. $\begin{array}{l} 40 - \frac{1}{2}gt^2 \\ \underline{50 - \frac{1}{2}gt^2} \end{array}$	13. $\begin{array}{l} 2z + 4x + y \\ \underline{2x + \quad y - z} \end{array}$	17. $\begin{array}{l} 4a + 2b - 9 \\ \underline{8c + 4a - 6d} \end{array}$
10. $\begin{array}{l} 5x + 7y - 8z \\ \underline{3x + \quad y - 4z} \end{array}$	14. $\begin{array}{l} 6c + 3d + e \\ \underline{3c + 2d \qquad} \end{array}$	18. $\begin{array}{l} 2x^4 + 5x \quad - 1 \\ \underline{3x^3 - 7x^2 + 8x} \end{array}$

19. A broker had $7a + 5b$ dollars in a bank and withdrew $a + 4b$ dollars. How much did he still have in the bank?

20. A coal dealer bought c carloads of coal containing 40 tons each, and d carloads of 50 tons each; he sold $5c$ tons to one customer, $8d$ tons to another, and $12c$ tons to another. How many tons had he left?

90. Removal of Parentheses.

1. If a parenthesis is preceded by the sign $+$, the terms within the parenthesis are to be added to what precedes, hence the parenthesis may be removed without altering the value of the expression.

For example:

$$a + (b + c) = a + b + c.$$
$$a + (b - c) = a + b - c.$$
$$a + (-b - c) = a - b - c.$$

2. If a parenthesis is preceded by the sign $-$, the terms within the parenthesis are to be subtracted from what precedes; hence the parenthesis may be removed provided the sign of each term within the parenthesis is changed, each sign $+$ to the sign $-$, and each sign $-$ to the sign $+$. Sec. 86.

For example:

$$a - (b + c) = a - b - c.$$
$$a - (b - c) = a - b + c.$$
$$a - (-b - c) = a + b + c.$$

WRITTEN EXERCISES

Remove parentheses and unite terms when possible:

1. $a+(b-3a)$.
2. $5-(t+2)$.
3. $3a-(2a+b)$.
4. $7b-(4a-6b)$.
5. $a-(2b-5a)-4b$.
6. $11t+(-3t-1)$.
7. $4x+7y-(3x+2y)$.
8. $d+3d^2-(2d-d^2)$.
9. $5-3p+(-18+2p)$.
10. $a-bx-(2a+bx)$.
11. $7q+5-(-11-3q)$.
12. $-(-3x-2y+11z)$.
13. $-(4a+3b-6c)$.
14. $x-3x^2+7-(2x^2+5-3x)$.
15. $9t+3-(2t-1)$.
16. $5x-12y-(3x+2y)$.
17. $4a+7b-(2a+3b)$.
18. $7-6m-(1+3m)$.
19. $11p+1-(-p+3)$.
20. $15x^2+7x-(18x^2-3x)$.
21. $4m^3-7m^2+3m-(-2m^3-9m^2)$.
22. $a^2-5ab+7ac+b^2-(4ab+7a^2-6b^2)$.
23. $x^2+3xy-4xz+7y^2-(z^2-3xz-4x^2-y^2)$.

24. Calculate the value of $A-(2B-C)$ when

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$A=$	24	1	-4	x	$4d$	$a+x$	$p+3$
$B=$	3	-5	3	$2x$	$-6d$	$3a-x$	$q+5$
$C=$	8	9	-5	$3x$	$3d$	$4a+7x$	$2p+3q$

Solve the equations:

25. $2x-(2+x)=6$.
26. $\frac{1}{2}x+(2-x)=6$.
27. $4x-(2x+1)=5$.
28. $5-(4x+2)=4$.
29. $7-(4x+3)=0$.
30. $6y+(4y-10)=10$.
31. $12z-(2+6z)=16$.
32. $2\frac{1}{2}+(2\frac{1}{2}-5t)=0$.

91. The methods of Sec. 90 may be applied when there is a parenthesis within a parenthesis. In this case the signs, $\{\ \}$, $[\ ]$, $\overline{\qquad}$, are commonly used to distinguish the different

parentheses. They may be removed one at a time, usually the inner one first, although it is likewise possible to begin with the outer one.

For example:

Beginning with the inner parenthesis,

$$\begin{aligned} 5a-\{6a+3b-(2a-5b)\} &= 5a-\{6a+3b-2a+5b\} \\ &= 5a-\{4a+8b\} \\ &= 5a-4a-8b \\ &= a-8b. \end{aligned}$$

Beginning with the outer parenthesis,

$$\begin{aligned} 5a-\{6a+3b-(2a-5b)\} &= 5a-6a-3b+(2a-5b) \\ &= -a-3b+2a-5b \\ &= a-8b. \end{aligned}$$

When the first parenthesis is removed the signs within the second one are not changed because the expression is taken as a single term.

WRITTEN EXERCISES

Remove parentheses and unite terms as much as possible:

1. $4x-\{3x-(2+x)\}$.
2. $7+\{4-(5x+2)+3x\}$.
3. $2+(5a-\overline{3a+4a})+6a$.
4. $x^3-\{3x^2-(2x^3+1)\}$.
5. $-\{4x^2+(3x-\overline{5x^2-9x})\}$.
6. $a-3b-\{b-3a+(3b-a)\}$.

92. Introduction of Parentheses.

The value of a polynomial is not changed:

1. If any number of terms with their *signs unchanged* are grouped in a parenthesis preceded by the *sign* $+$.

2. If any number of terms with their *signs changed* are grouped in a parenthesis preceded by the *sign* $-$. Sec. 90.

For example:

$$2a+5b-6c=2a+(5b-6c).$$
$$2a+5b-6c=2a-(-5b+6c).$$
$$4a-7m+2x=4a-(7m-2x).$$

93. The fact that the terms of a polynomial may be grouped as stated in Sec. 92, 1, without changing the value of the polynomial, is called the **Associative Law of Addition.**

WRITTEN EXERCISES

Write as a plus a parenthesis:

1. $a+6b-4c.$ 3. $a-4b-c+5d.$ 5. $a+9b-7c+3d.$

2. $a-3b+7c.$ 4. $a-8b+c-1.$ 6. $a+2b+7c+4.$

7–12. Write each of the expressions in Exercises 1–6 as a minus a parenthesis.

13–18. In each of the expressions in Exercises 1–6 place the terms involving b and c in a parenthesis preceded by the sign $-$.

94. The sum or the difference of terms alike with respect to certain letters may be indicated by the use of parentheses.

EXAMPLES

1. Add ax and bx.

$$\begin{array}{lr} \text{Addend} & ax \\ \text{Addend} & \underline{bx} \\ \text{Sum} & (a+b)x \end{array}$$

2. Subtract $(c+n)xy$ from $(b+2c)xy$.

$$\begin{array}{r} (b+2c)\,xy \\ \underline{(c+\ \ n)\,xy} \\ (b+c-n)\,xy \end{array}$$

WRITTEN EXERCISES

Add:

1. $\begin{array}{r} (a+b)x \\ \underline{bx} \end{array}$ 3. $\begin{array}{r} (a+c)xy \\ \underline{(a-c)xy} \end{array}$ 5. $\begin{array}{r} (a+c)(m^2+n) \\ \underline{c(m^2+n)} \end{array}$

2. $\begin{array}{r} (1+2b)x^2 \\ \underline{-bx^2} \end{array}$ 4. $\begin{array}{r} (a+b)(p+q) \\ \underline{(c+d)(p+q)} \end{array}$ 6. $\begin{array}{r} (m-14+p)x^2y \\ \underline{(-m+2n-p)x^2y} \end{array}$

Subtract:

7. $\begin{array}{r} (a+2b)x^p \\ \underline{bx^p} \end{array}$ 8. $\begin{array}{r} (2m-4)x^2y \\ \underline{(37-n)x^2y} \end{array}$ 9. $\begin{array}{r} (a+c)(m^2+n) \\ \underline{(2a+c)(m^2+n)} \end{array}$

10. $\begin{array}{r} (a + c)x^y \\ \underline{(a+2c)x^y} \end{array}$ 11. $\begin{array}{r} (a+c)(x+y) \\ \underline{2c(x+y)} \end{array}$ 12. $\begin{array}{r} (1+b+c)p^2q^2 \\ \underline{(a-2b-2)p^2q^2} \end{array}$

SUMMARY

I. **Definitions.**

1. *Subtraction* is the process of finding the difference between two numbers called the minuend and the subtrahend. Sec. 83.

2. The *difference* is the number which added to the subtrahend makes the minuend. Sec. 84.

3. The fact that the terms of a polynomial may be grouped as stated in 6 below without changing the value of the expression is called the *Associative Law of Addition.* Sec. 93.

II. **Processes.**

1. To subtract a negative number, find the number which added to it makes the minuend. Sec. 85.

2. Subtraction may always be performed by adding to the minuend the subtrahend with its sign changed. Sec. 86.

3. To subtract a polynomial arrange its terms under the like terms of the minuend, subtract each term from the one above it, and use the signs obtained as the signs of the result. Sec. 88.

4. A parenthesis preceded by the sign + may be removed without altering the value of the expression. Sec. 90.

5. A parenthesis preceded by the sign – may be removed by changing the signs of the terms in the parenthesis without altering the value of the expression. Sec. 90.

6. Any number of terms with their signs unchanged may be grouped in a parenthesis preceded by the sign + without changing the value of the expression. Sec. 92.

7. Any number of terms may be grouped in a parenthesis preceded by the sign – without changing the value of the expression, provided that the sign of every term placed in the parenthesis is changed. Sec. 92.

REVIEW

ORAL EXERCISES

Subtract:

1. $\begin{array}{r} 5n \\ \underline{16n} \end{array}$
2. $\begin{array}{r} 7a \\ \underline{-9a} \end{array}$
3. $\begin{array}{r} 11a+10b \\ \underline{a+\ \ 3b} \end{array}$
4. $\begin{array}{r} x+y \\ \underline{x-y} \end{array}$
5. $\begin{array}{r} a-7 \\ \underline{2a-3} \end{array}$
6. $\begin{array}{r} a-x \\ \underline{a-1} \end{array}$
7. $\begin{array}{r} 3a-b \\ \underline{3a+b} \end{array}$
8. $\begin{array}{r} 7a-5b \\ \underline{a-2b} \end{array}$
9. $\begin{array}{r} 3x-5y \\ \underline{x-\ \ y} \end{array}$
10. $\begin{array}{r} p^2-q^2 \\ \underline{p^2+q^2} \end{array}$
11. $\begin{array}{r} x^2+y \\ \underline{2x^2-y} \end{array}$
12. $\begin{array}{r} ax-b \\ \underline{bx-c} \end{array}$

WRITTEN EXERCISES

Remove the parentheses:

1. $(a+b-c)+(a-b+c)$.
2. $(a+b-c)-(a-b+c)$.
3. $7a-3b-(5a+3b)$.
4. $3x-7-(9x-11)$.
5. $2y-(\frac{1}{2}tx+\frac{1}{3}y)$.
6. $(\frac{1}{3}tx-\frac{1}{2}y)-(\frac{1}{6}y-\frac{1}{6}tx)$.
7. $m-[(a-b)-(c-m)]$.
8. $m+[(a-b)+(b+d)]$.
9. $6x+5y-32-(5x-3y+22)$.
10. $6a^2-(3ab+2ac)-(2ac+3ab)$.

Subtract and test:

11. $\begin{array}{r} 3x^2+2x^2y^2+z^2 \\ \underline{x^2-2x^2y^2\qquad} \end{array}$
12. $\begin{array}{r} m-3n+p-7 \\ \underline{m-4n-p+8} \end{array}$
13. $\begin{array}{r} 6c+3d+a-3z \\ \underline{3c-2d-2a\qquad} \end{array}$
14. $\begin{array}{r} 4x+3y-4z+8 \\ \underline{-7x-3y-2z+17} \end{array}$
15. $\begin{array}{l} 8x^3-9x^2+\ \ x^4-x+16+x^5 \\ \underline{2x-7\ \ +5x^4-x^2-x^3+6x^5} \end{array}$
16. $\begin{array}{r} 5.65a+\ 7\frac{3}{5}b-27\frac{3}{4}c+.76x-\ \ 1\frac{1}{6}y \\ \underline{4\frac{1}{4}a-9.38b+2.65c-13\frac{1}{5}x-0.375y} \end{array}$

17. A broker bought 10 m bu. of wheat, 6 n bu. of corn, and 5 p bu. of barley, and sold 5 m bu. of wheat, 2 n bu. of corn, and p bu. of barley. How many bushels of grain had he left?

CHAPTER VII

MULTIPLICATION

MULTIPLICATION OF MONOMIALS

95. PREPARATORY.

1. $5 \cdot 3¢ = (\ )¢.$ 3. $5 \cdot 3$ ft. $= (\ )$ft. 5. $5 \cdot 3f = (\ )f.$

2. $6 \cdot 3d = (\ )d.$ 4. $7 \cdot 5x = (\ )x.$ 6. $6 \cdot 6abc = (\ )abc.$

7. $3a \cdot 4b = 3 \cdot 4ab = ?$ 8. $\frac{1}{2}a \cdot 8b = \frac{1}{2}$ of $8ab = ?$

9. $4x \cdot 6y \cdot \frac{3}{8}z = 4 \cdot 6 \cdot \frac{3}{8}xyz = (\quad)xyz.$

96. To find the product of two monomials, *take the product of their literal parts for the literal part of the product, and the product of their coefficients for the coefficient of the product.*

97. Repeated Factors. If the same letter occurs more than once as a factor in the product, it should be written only once, with the proper exponent (Sec. 32, p. 17).

For example:

$$3a \cdot 2ab = 6aab = 6a^2b.$$
$$a^2b^3 \cdot ab^2 = aabbb \cdot abb = aaa \cdot bbbbb = a^3b^5.$$

98. Involution. Repeating a number as a factor is called *raising it to a power,* or **involution.**

99. Law of Exponents in Multiplication. According to the above examples, *the exponent of any letter in a product is the sum of the exponents of that letter in all of the factors.*

In symbols, this law is expressed:

$$a^m \cdot a^r = a^{m+r}.$$

ORAL EXERCISES

State the products:

1. $t^2 \cdot t^4$.
2. $y^5 \cdot y^2$.
3. $x^2y \cdot y^3$.
4. $8\,c \cdot 8\,c$.
5. $7\,a \cdot 6\,ab$.
6. $a^3 \cdot a^2$.
7. $4\,a \cdot 2\,ax$.
8. $4\,x^3 \cdot 2\,x^2$.
9. $a^2x \cdot a^3x^4$.
10. $12\,ab \cdot 2\,ab^2$.
11. $3\,xy^2 \cdot 5\,x^2y$.
12. $2\,a^4 \cdot a^5$.
13. $(4\,p^2)(8\,p^4)$.
14. $(am^3)(am^3)$.
15. $2\,a^3r \cdot 6\,a^2r^5$.
16. $a^2 \cdot a^2 \cdot a^2$.
17. $v^4 \cdot v^3 \cdot v^5$.
18. $3\,x \cdot x^3 \cdot x^2$.

WRITTEN EXERCISES

Multiply as indicated:

1. $48\,ab^3 \cdot 10\,bc^2$.
2. $24\,a^2bc \cdot 12\,c$.
3. $214\,b \cdot 6\,c^3 \cdot c^2$.
4. $16\,a^2 \cdot b \cdot 15\,ac$.
5. $36\,x^2 \cdot 4\,y \cdot 3\,xz^2$.
6. $12\,m \cdot 16\,mr \cdot r^3$.
7. $3\,x^2 \cdot 4\,ax^3 \cdot 8\,bx^5$.
8. $5\,a^3 \cdot 4\,ab \cdot 7\,bc^3$.
9. $27\,a^2 \cdot 63\,b^2 \cdot 4\,ab$.

100. PREPARATORY.

1. A man earned \$3 on Monday and \$3 on Tuesday. How many dollars did he earn in the two days?

2. To multiply 3 by 2 is to take 3 how many times as the addend?

3. A man earned a dollars per day for b days. How much did he earn in all?

4. To multiply a by b is to take a how many times as an addend?

5. A man lost \$3 on Monday and \$3 on Tuesday. How many dollars did he lose in the two days?

6. To multiply -3 by 2 is to take -3 how many times as an addend?

7. A man lost a dollars per day for b days. How much did he lose in all?

8. To multiply $-a$ by b is to take $-a$ how many times as an addend?

101. Multiplication by Relative Numbers. *Multiplication by a positive integer means taking the multiplicand as an addend as many times as there are units in the multiplier.*

Correspondingly, *multiplication by a negative integer means taking the multiplicand as a subtrahend as many times as there are units in the multiplier.*

For example:

4 multiplied by $-3 = -4 - 4 - 4 = -12$.

-4 multiplied by $-3 = -(-4) - (-4) - (-4) = +4 + 4 + 4 = 12$.

a multiplied by $-b = -a - a \cdots b$ times $= -ab$.

$-a$ multiplied by $-b = -(-a) - (-a) \cdots b$ times $= +ab$.

102. The law of signs in multiplication, which applies to integral and fractional numbers alike, may be stated thus: If both factors are positive or if both are negative, their product is *positive*. If one is positive and the other negative, their product is *negative*.

The numerical value of the product is the product of the numerical value of the factors.

In symbols:

$$+a \text{ times } +b = +ab$$
$$-a \text{ times } +b = -ab$$
$$+a \text{ times } -b = -ab$$
$$-a \text{ times } -b = +ab$$

This law is easily remembered in the form:

The product of two factors of like signs is positive and of two factors of unlike signs is negative.

ORAL EXERCISES

State the product in each of the following:

1. $5 \cdot 3$.	7. $x(-y)$.	13. $ax \cdot x$.	19. $(-q)^2$.
2. $-5 \cdot 3$.	8. $-x(-y)$.	14. $-ax \cdot x$.	20. $p(-p)$.
3. $5(-3)$.	9. $ax \cdot b$.	15. $-ax \cdot x$.	21. $(-5)^2$.
4. $-5(-3)$.	10. $-ax \cdot b$.	16. $-ax(-x)$.	22. $2mv(-v)$.
5. $x \cdot y$.	11. $ax(-b)$.	17. $2mv \cdot v$.	23. $-2mv(-v)$.
6. $-x \cdot y$.	12. $-ax(-b)$.	18. $-2mv \cdot v$.	24. $(-p)(-p)$.

WRITTEN EXERCISES

Copy and supply the products:

1. $\begin{array}{r}-a\\ \underline{2}\end{array}$	8. $\begin{array}{r}3y^2\\ \underline{-6xy}\end{array}$	15. $\begin{array}{r}-\pi r^2\\ \underline{\frac{4}{3}\pi r}\end{array}$	22. $\begin{array}{r}.9xy^2\\ \underline{.1x^2y}\end{array}$
2. $\begin{array}{r}-ab\\ \underline{2c}\end{array}$	9. $\begin{array}{r}-5x^2y\\ \underline{2xy^2}\end{array}$	16. $\begin{array}{r}+m^2q\\ \underline{-mq^2}\end{array}$	23. $\begin{array}{r}-.3x^2y\\ \underline{\frac{1}{3}xy}\end{array}$
3. $\begin{array}{r}-x^2y\\ \underline{xy^2}\end{array}$	10. $\begin{array}{r}6mn\\ \underline{4m^2n}\end{array}$	17. $\begin{array}{r}-fs\\ \underline{fs}\end{array}$	24. $\begin{array}{r}-.4pq\\ \underline{-4rs}\end{array}$
4. $\begin{array}{r}-2ab\\ \underline{3c}\end{array}$	11. $\begin{array}{r}-abc\\ \underline{-abc^2}\end{array}$	18. $\begin{array}{r}-ab\\ \underline{-cd}\end{array}$	25. $\begin{array}{r}-.5uv\\ \underline{-.5vw}\end{array}$
5. $\begin{array}{r}-x^2y\\ \underline{xy}\end{array}$	12. $\begin{array}{r}-4ab^2\\ \underline{-3a^2b}\end{array}$	19. $\begin{array}{r}+nm\\ \underline{-lr}\end{array}$	26. $\begin{array}{r}-5ab\\ \underline{-5ab}\end{array}$
6. $\begin{array}{r}x^2y\\ \underline{-xy^2}\end{array}$	13. $\begin{array}{r}-a^2bc\\ \underline{-abc^2}\end{array}$	20. $\begin{array}{r}+gt\\ \underline{-\frac{1}{2}t}\end{array}$	27. $\begin{array}{r}-4ax^3\\ \underline{-4ax^3}\end{array}$
7. $\begin{array}{r}-5xy\\ \underline{-x^2}\end{array}$	14. $\begin{array}{r}\frac{1}{2}mv\\ \underline{v}\end{array}$	21. $\begin{array}{r}+\frac{3}{4}ab^2\\ \underline{-\frac{1}{3}a^2b}\end{array}$	28. $\begin{array}{r}-7dt^2\\ \underline{-7dt^2}\end{array}$

Multiply:

29. $-acx$, $-2axy$.
30. a^2xy, $-cx$.
31. ax^2, $-5bx^2$.
32. $6ab^2$, $3a^2b$.
33. $2a^5$, $-3b^4$.
34. $-4c$, $-6ab$.
35. $2x^2z^2$, $-3x^4z^2y$.
36. $-5a^2z^2$, $-2x^2y^3$.
37. $2a^2b^2c^2$, $3abc^3$.
38. $-a$, $-a^3$.
39. $-a^2b^2$, $-a^2c^2$.
40. $-4a^2xn$, $\frac{1}{2}ax^2$.
41. $-mn^2p$, $-15na^2q$.
42. $12m^2n$, $-\frac{1}{2}mn^3$.
43. mn^2p, $-15m^3np$.
44. $-5a^2b^2c$, $2ab^3c^2$.
45. $10p^2q^2r^2t$, $-\frac{1}{5}pq^2rt^3$.
46. $12xyz^2$, $-\frac{3}{4}xy^2z$.

103. To find the product of several monomials, *multiply the product of the first two by the third, that product by the fourth, and so on.*

104. The three steps in this process are:

1. Find the sign of the product.

The sign is plus when the number of negative factors is even, and minus when the number of negative factors is odd.

2. Find the numerical coefficient of the product.

3. Find the literal part of the product.

EXAMPLE

Find the product of $-2a$, $+3b$, $-2ab$, $-4bc$.

1. The sign is $-$.
2. $2 \cdot 3 \cdot 2 \cdot 4 = 48$.
3. $a \cdot b \cdot ab \cdot bc = a^2b^3c$.
4. $\therefore -2a \cdot 3b \cdot -2ab \cdot -4bc = -48a^2b^3c$.

ORAL EXERCISES

1. $a(-b) = ?$ $-ab \cdot c = ?$ $a \cdot -b \cdot c = ?$
2. $-a \cdot b = ?$ $-ab \cdot c = ?$ $-a \cdot b(-c) = ?$
3. $-a(-b)c = ?$ $-a(-b)c(-d) = ?$
4. $-a(-b)(-c) = ?$ $2a(-2b)3c = ?$ $a(-3b)(-4c) = ?$
5. $-x \cdot y(-z) = ?$ $3x(-2y)(-z) = ?$ $x(-6y)3z = ?$

105. *If any factor is zero, the product is zero.*

For example:

$3 \cdot 0 = 0$; similarly, $-x \cdot 0 = 0$, and $x \cdot y \cdot z \cdot 0 = 0$.

WRITTEN EXERCISES

Multiply:

1. $6, -3, 5$.
2. $-7, 3, -1$.
3. $a, 0, -c$.
4. $-6, -0, -4$.
5. $2x, -3y, -2z$.
6. $ax, -bx, -cx$.
7. $-a, -a, 0$.
8. $(-q), (-q), (-q)$.
9. $(-6), (-6)$.
10. $(-4x), (-4x)$.
11. $(-2)^3, (-a)^3$.
12. $x, (-x)^2, (-x)^3$.
13. $x^3, (-x)^3$.
14. $-x^2, (-x)^2$.
15. $(-2)^2, (-3)^3, (-1)^2$.
16. $2gx, 4gx, -gx^6$.
17. $(-3ab)^3, 5, (-3ab)^2$.
18. $m^2, -n^2, (-p)^2$.

19. $-p^3, -q^3, pqr, (-p)^4$.

20. $-x, (-x)^4, (-y)^3, xyz$.

21. $-a^3, -2a, 3a, -4a^2$.

22. $-6, -\frac{3}{8}, 0, -\frac{1}{2}$.

23. $ax^3, -bx^4, abx^5, -bx^2$.

24. $-a, -a^2, -a^3, -a^4$.

25. Suppose $12a^2b^3c$ to be taken as the product of $4ab^2$ and $3a^2c$. Test this product by letting a, b, and c each equal 1.

26. Also test it by letting $a = 2$, $b = 3$, and $c = 5$. Why did not the test applied in Exercise 25 reveal the error?

27. Test the work in Exercises 13–20 above.

106. Signs of Factors. It follows from Sec. 102 that, if the product of two factors is positive, the factors must have like signs; but, if the product is negative, the factors must have unlike signs.

For example:

$$8ax = 2a \cdot 4x, \text{ or } (-2a)(-4x).$$
$$-14b^2c = (7b)(-2bc), \text{ or } (-7b)(2bc).$$

WRITTEN EXERCISES

Write a set of two factors for each of the following:

1. b^2h.	6. πr^2h.	11. $15ax^2y^2$.	16. $14y^2$.
2. $\frac{1}{2}gt^2$.	7. $\frac{4}{3}\pi r^3$.	12. $-7mn^2p$.	17. $-48x$.
3. mx^2.	8. $\frac{1}{2}mv^2$.	13. $16ab^2c$.	18. $21a^3x^3$.
4. $-\pi r^2$.	9. $\frac{1}{6}\pi d^3$.	14. $-8a^3b$.	19. $-md^3$.
5. $4\pi r^2$.	10. $6ax^2$.	15. $14br^2$.	20. $2\pi rh$.

MULTIPLICATION OF POLYNOMIALS

107. To multiply a polynomial by a monomial *multiply each term of the multiplicand by the monomial and use the signs obtained as the signs of the product.*

For example:

$$\begin{array}{r} 2a - 3b + 5c \\ 6a \\ \hline 12a^2 - 18ab + 30ac \end{array}$$

108. Distributive Law. The fact that a polynomial is multiplied by multiplying each of its terms separately and taking

the algebraic sum of the partial products thus found is called the **Distributive Law of Multiplication.**

The formula, $a(b+c) = ab + ac$, expresses this law in symbols.

ORAL EXERCISES

Read and supply the numbers for the blanks:

1. $$\begin{array}{l} 3a+4b \\ \underline{3\qquad\quad} \\ (\ )a+(\ )b \end{array}$$

2. $$\begin{array}{l} 5a^2+2b \\ \underline{10\qquad\quad} \\ (\ )a^2+(\ )b \end{array}$$

3. $$\begin{array}{l} 10x+y \\ \underline{5\qquad\quad} \\ (\ )x+(\ )y \end{array}$$

4. $$\begin{array}{l} 3a+4b \\ \underline{3a\qquad\quad} \\ (\ )a^2+(\ )ab \end{array}$$

5. $$\begin{array}{l} 10m+2n \\ \underline{5mn\qquad\quad} \\ (\ )m^2n+(\ )mn^2 \end{array}$$

6. $$\begin{array}{l} 4x^2y+xy^2 \\ \underline{3xy\qquad\quad} \\ (\ )x^3y^2+(\ )x^2y^3 \end{array}$$

7. $$\begin{array}{l} 6a^2+2b \\ \underline{3b\qquad\quad} \\ 18(\ )+6(\ ) \end{array}$$

8. $$\begin{array}{l} 5m+3p^2 \\ \underline{3mp\qquad\quad} \\ (\ )m^2p+(\ )mp^3 \end{array}$$

9. $$\begin{array}{l} 9x^2+y^2 \\ \underline{5xy\qquad\quad} \\ 45(\ )+5(\ ) \end{array}$$

109. To test the work of multiplication, use arbitrary values as in Sec. 78, p. 46. The product of the values of the multiplicand and the multiplier must equal the value of the product.

Unity is the easiest number to substitute for the letters; it tests the coefficients and signs in the work of multiplication. It does not, however, test the exponents (see Exercises 25 and 26, p. 68); but this does not impair the test materially, since errors in exponents alone seldom occur.

WRITTEN EXERCISES

Multiply and test:

1. $$\begin{array}{l} ax+4 \\ \underline{3\qquad\quad} \end{array}$$

2. $$\begin{array}{l} a+b \\ \underline{c\qquad\quad} \end{array}$$

3. $$\begin{array}{l} 2a+3b \\ \underline{4c\qquad\quad} \end{array}$$

4. $$\begin{array}{l} 7ax+3a \\ \underline{5x\qquad\quad} \end{array}$$

5. $$\begin{array}{l} 8a^2+2b^2 \\ \underline{ab\qquad\quad} \end{array}$$

6. $$\begin{array}{l} 7ax+3bx^2 \\ \underline{cx\qquad\quad} \end{array}$$

7. $$\begin{array}{l} a^2+2bx^3 \\ \underline{3b^2x\qquad\quad} \end{array}$$

8. $$\begin{array}{l} 3m^2+r^2 \\ \underline{4a\qquad\quad} \end{array}$$

9. $$\begin{array}{l} mt^2+v \\ \underline{vt\qquad\quad} \end{array}$$

10. $4x^2(a-5x)$.
11. $a(2x^3+1)$.
12. $a^2b(a^2c-b^2d)$.
13. $2x(3x+5y)$.
14. $(7x-5)2a$.
15. $(4x+5t)2tx$.
16. $6q^2(5q+18q^3)$.
17. $\frac{1}{2}r(2r^2+\frac{1}{4}r)$.
18. $6t(t^2+\frac{1}{2}at)$.

110. Removal of Parentheses. If a *parenthesis used to indicate multiplication is removed, the multiplication must be performed.*

Thus: $7a-5(9a-4b)=7a-45a+20b=-38a+20b.$

WRITTEN EXERCISES

Remove parentheses and unite terms as much as possible:

1. $3+5(6-4)$.
2. $5x+3(11x-5)$.
3. $7(4a-2b)+10b$.
4. $6y-7(4y+3t)$.
5. $11-3(7-2x)$.
6. $4a-12(7-6a)$.
7. $9(a-x)-a(5+x)$.
8. $-5(2x-1)+3(4x-8)$.
9. $a(b+c)-c(a+b)+b(a-c)$.
10. $2m-\{4m+7(6m-1)\}$.
11. $p\{4r-3r(1-a)+5ar\}$.
12. $x^2\{x^2-2a(2x-3a)\}-a^3(4x-a)$.
13. $-10\{x-6[x-(y-z)]\}+60\{y-(z+x)\}$.
14. From

$$[m(3m-p)-2n(4n-3p)]x+[m(p-m)-p(2n+p)]y$$

take

$$3\left[p\left(2n-\frac{3p}{2}\right)-\frac{p}{2}(2n-3p)\right]x-[p(p-m)+2n(2n+p)]y$$

and then find the value of the result for $x=n=-p=1$, and $y=m=0.2$.

Remove the parentheses and solve the equations:

15. $x(x-3)+1-x(x-5)=0$.
16. $x^2+3-x(x+4)=16$.
17. $x^2(x-1)-x^3+x^2+2x=12$.

111. The multiplication of literal numbers is similar to the multiplication of numbers expressed by figures.

For example:

MULTIPLICATION WITH FIGURES	MULTIPLICATION WITH LETTERS	TEST
32	$3a+2b$	5
14	$a+4b$	5
$128=4\times 32$	$3a^2+2ab \quad =a(3a+2b)$	
$320=10\times 32$	$12ab+8b^2=4b(3a+2b)$	
$448=14\times 32.$	$3a^2+14ab+8b^2=(a+4b)(3a+2b)$	25

112. To multiply by a polynomial, *multiply by each term of the polynomial, add like terms, and use the signs obtained as the signs of the result.*

Thus:

$$\begin{array}{r}
2a^2+5a-2 \\
a^2-3a+1 \\
\hline
2a^4+5a^3-2a^2 \qquad\qquad \\
-6a^3-15a^2+6a \quad \\
2a^2+5a-2 \\
\hline
2a^4-a^3-15a^2+11a-2
\end{array}$$

WRITTEN EXERCISES

Multiply and test:

1. $\begin{array}{l} a+b \\ a+b \\ \hline \end{array}$

2. $\begin{array}{l} a+b \\ a-b \\ \hline \end{array}$

3. $\begin{array}{l} a-b \\ a-b \\ \hline \end{array}$

4. $\begin{array}{l} c+1 \\ c-1 \\ \hline \end{array}$

5. $\begin{array}{l} x+2 \\ x+2 \\ \hline \end{array}$

6. $\begin{array}{l} z^2+5 \\ z^2+5 \\ \hline \end{array}$

7. $\begin{array}{l} 2a-1 \\ 2a-1 \\ \hline \end{array}$

8. $\begin{array}{l} 12+x \\ 12-x \\ \hline \end{array}$

9. $\begin{array}{l} 3y-5 \\ 2y+4 \\ \hline \end{array}$

10. $\begin{array}{l} x+a \\ x+b \\ \hline \end{array}$

11. $\begin{array}{r} 3a+x \\ a+b \\ \hline \end{array}$

12. $\begin{array}{r} 4a+5 \\ x-a \\ \hline \end{array}$

13. $\begin{array}{r} m+3 \\ 3m+2 \\ \hline \end{array}$

14. $\begin{array}{l} m^2-n^2 \\ m^2+n^3 \\ \hline \end{array}$

15. $\begin{array}{l} x^2+1 \\ x^2-1 \\ \hline \end{array}$

16. $\begin{array}{r} 2a+b \\ a+2b \\ \hline \end{array}$

17. $\begin{array}{r} 2a-b \\ c-3a \\ \hline \end{array}$

18. $\begin{array}{l} 3x+2y \\ 2x+3y \\ \hline \end{array}$

19. $\begin{array}{l} 3ab+4b^2 \\ 2ab-3b^2 \\ \hline \end{array}$

20. $\begin{array}{l} x^2+3x-1 \\ x+3 \\ \hline \end{array}$

21. $\begin{array}{l} x^2-4x+3 \\ x-2 \\ \hline \end{array}$

22. $\begin{array}{l} x^2-ax+b \\ x-c \\ \hline \end{array}$

23. $\begin{array}{l} x^2-ax+b \\ 3x+a \\ \hline \end{array}$

24. $\begin{array}{l} t^2+tu+u^2 \\ t-u \\ \hline \end{array}$

25. $\begin{array}{l} a+b-c \\ a-b+c \\ \hline \end{array}$

26. $\begin{array}{r} 3ax-4by+1 \\ 2a-3b-4 \\ \hline \end{array}$

27. $(a+b-3)^2$.

28. $(2a-5x-4y)^2$.

29. $\begin{array}{l} x^2+y^2-z^2 \\ x+y-z \\ \hline \end{array}$

30. $\begin{array}{l} 2a-b+3c \\ 2a+b-3c \\ \hline \end{array}$

31. $\begin{array}{l} xy+yz+xz \\ x-y+z \\ \hline \end{array}$

113. ***To find the product of expressions involving literal coefficients and exponents, find the product of the coefficients and add the exponents as in numerical cases.***

EXAMPLES

1. Multiply ax and $(a-b)x$.

$$\begin{array}{r} (a-b)x \\ ax \\ \hline a(a-b)x^2 \end{array}$$

2. Multiply x^n+1 by x^n-2.

$$\begin{array}{r} x^n+1 \\ x^n-2 \\ \hline x^{2n}-x^n-2 \end{array}$$

WRITTEN EXERCISES

Multiply:

1. $(5a+x)(5a-x)$.
2. $(x-y)(x^m-y^m)$.
3. $(x^m+y^n)(x+y)$.
4. $(4x^m+3y^n)(4x^m-3y^n)$.
5. $(x^m-y^n)(x^{2m}-y^{2n})$.
6. $(a^{2n}+c^{2n})(a^{mn}+c^{mn})$.
7. $z^n(z^r+z^s)$.
8. $(xc-1)cx$.
9. $(a-b)y \cdot cby$.
10. $(c+d)x \cdot (c-d)x$.
11. $(a^m+c)(a^m-c)$.
12. $(a-3ab^3)(a+3ab^3)$.

SUMMARY

I. Definitions and Laws.

1. *Involution* is repeating a number as a factor. Sec. 98.

2. The law of exponents in multiplication is $a^m \cdot a^r = a^{m+r}$. Sec. 99.

3. The law of signs in multiplication is:
The product of two factors of like signs is positive and of unlike signs is negative. Sec. 102.

4. The fact that an algebraic sum is multiplied by multiplying each of its terms separately and taking the algebraic sum of the partial products thus found is called the *Distributive Law of Multiplication.* Sec. 108.

The formula $a(b+c)=ab+ac$ expresses this law in symbols.

II. **Processes.**

1. *To find the product of two monomials,* take the product of their literal parts for the literal part of the product, and the product of their coefficients for the coefficient of the product. Sec. 96.

2. The product of two factors with like signs is positive and of two factors with unlike signs is negative. Sec. 102

3. *To find the product of several monomials,* multiply the product of the first two by the third, that product by the fourth, and so on. Sec. 103.

4. In finding the product of several factors:

(1) Find the sign of the product.

(2) Find the numerical coefficient of the product.

(3) Find the literal part of the product. Sec. 104.

(4) If any factor is zero, the product is zero. Sec. 105.

5. *To multiply a polynomial by a monomial,* multiply each term of the multiplicand by the monomial and use the signs obtained as the signs of the product. Sec. 107.

6. *To test the work of multiplication,* use arbitrary values. The product of the values of the multiplicand and the multiplier must equal the value of the product. Sec. 109.

7. *To multiply by a polynomial,* multiply by each term of the polynomial, add like terms, and use the signs obtained as the signs of the result. Sec. 112.

REVIEW

ORAL EXERCISES

State the products:

1. $\begin{array}{r} 4x \\ \underline{9x} \end{array}$	3. $\begin{array}{r} 7a \\ \underline{4a} \end{array}$	5. $\begin{array}{r} -9m \\ \underline{3m} \end{array}$	7. $\begin{array}{r} -2ax \\ \underline{+5x^2} \end{array}$	9. $\begin{array}{r} 7x^2y \\ \underline{-\ \ xy} \end{array}$
2. $\begin{array}{r} 8y \\ \underline{-3y} \end{array}$	4. $\begin{array}{r} -6t \\ \underline{-2t} \end{array}$	6. $\begin{array}{r} 5ab \\ \underline{7ac} \end{array}$	8. $\begin{array}{r} -6ay \\ \underline{-4ay} \end{array}$	10. $\begin{array}{r} -9am \\ \underline{\frac{1}{3}ar} \end{array}$

WRITTEN EXERCISES

Multiply and test:

1. $\begin{array}{l} ax+3 \\ \underline{ax+5} \end{array}$

2. $\begin{array}{l} 4ab+c \\ \underline{2ab+3c} \end{array}$

3. $\begin{array}{l} x^2+13x-5 \\ \underline{x-2} \end{array}$

4. $\begin{array}{l} x^3-7x^2+5x-3 \\ \underline{2x-4} \end{array}$

5. $\begin{array}{l} x^2+3px-4p^2 \\ \underline{2x^2-7px-p^2} \end{array}$

6. $\begin{array}{l} a^3-3a^2y+3ay^2-y^3 \\ \underline{a-y} \end{array}$

7. $\begin{array}{l} x^2+xy+y^2 \\ \underline{x^2-xy+y^2} \end{array}$

8. $\begin{array}{l} 1-x+x^2 \\ \underline{1+x-x^2} \end{array}$

9. $\begin{array}{l} a+3x-1 \\ \underline{x+2a+1} \end{array}$

10. $\begin{array}{l} x^3+3x^2y+3xy^2+y^3 \\ \underline{x+y} \end{array}$

11. $\begin{array}{l} y+x+b \\ \underline{y+5x-b} \end{array}$

12. $\begin{array}{l} 5+4p+5q \\ \underline{1+4p-5q} \end{array}$

13. $\begin{array}{l} a-b-c \\ \underline{a-b-c} \end{array}$

14. $\begin{array}{l} 1+2x-x^2 \\ \underline{1+2x-x^2} \end{array}$

15. $(2a^n+3b^n)(2a^n+3b^n)$.

16. $(r^{p-1}-s^{p-1})(r-s)$.

17. $(4z^{3n}-2^{2n}+z^n-1)(3z^n+1)$.

18. $(3x^m-y^r)(3x^m+y^r)$.

19. $x^2y^s(x^ny^r-x^qy^4)$.

20. $(a-b)x\cdot(a+b)x$.

21. $ab\cdot(c-1)b$.

22. $ax^2\cdot(a-b)x^2\cdot(b+c)x^2$.

23. $\begin{array}{l} x-5 \\ \underline{x+6} \end{array}$

24. $\begin{array}{r} 2x+3 \\ \underline{x-1} \end{array}$

25. $\begin{array}{l} 3x+5 \\ \underline{2x-4} \end{array}$

26. $\begin{array}{r} 6t-3u \\ \underline{t+2u} \end{array}$

27. $\begin{array}{r} 3x^2+5 \\ \underline{x-9} \end{array}$

28. $\begin{array}{l} x^2+x+1 \\ \underline{x+1} \end{array}$

29. $\begin{array}{l} x^2-2x+1 \\ \underline{x-1} \end{array}$

30. $\begin{array}{l} x^2+3x+2 \\ \underline{x^2+x+2} \end{array}$

31. $\begin{array}{l} 2x^2-x+1 \\ \underline{2x-5} \end{array}$

32. $(x-1)(x-2)(x-3)$.

33. $(x+6)(x-5)(x-3)$.

34. $(a^3-a^2-1)(a+1)$.

35. $(m^2-m+1)(m+1)$.

36. $(x^4-x^2+1)(x^2+6)$.

37. $(x^2y-xy^2)(x+y)$.

38. $(m^5+m^3+1)(m^2+1)$.

39. $(y^4+y+2)(y^3-1)$.

40. $(y^3-3y+5)(y^2+10)$.

41. $(rs-r^2s^2)(rs+r^2s^2)$.

SUPPLEMENTARY WORK

When two polynomials can be arranged in descending powers of the same letter, the work of multiplication may often be shortened by *using the coefficients only.* This is called **multiplication by detached coefficients.**

Thus, to multiply $3x^2 - 2x + 1$ by $x - 1$:

WORK IN FULL	WORK WITH COEFFICIENTS ONLY	TEST
$$\begin{array}{l} 3x^2 - 2x + 1 \\ \underline{x - 1 } \\ 3x^3 - 2x^2 + x \\ \underline{ - 3x^2 + 2x - 1} \\ 3x^3 - 5x^2 + 3x - 1 \end{array}$$	$$\begin{array}{l} 3 - 2 + 1 \\ \underline{1 - 1 } \\ 3 - 2 + 1 \\ \underline{ - 3 + 2 - 1} \\ 3 - 5 + 3 - 1 \end{array}$$	$$\begin{array}{r\|l} x = 1 & 2 \\ & \underline{0} \\ & \\ & \\ & 0 \end{array}$$

The processes in the two cases are identical with the exception of the omission of the letters in the second case. The product of $3x^2$ and x, the first terms of the factors, shows that the first term of the product is $3x^3$, the next term must contain x^2 and the next x, because the terms in the result must be in order of degree. Thus we may perform the operation with coefficients, and then supply the proper letters and exponents.

When some powers of the letters are missing, zeros must be supplied as coefficients of the missing powers to keep a record of the places in which the powers are missing.

Thus, to multiply $2x^3 - 7x + 3$ by $2x - 5$, write $2x^3 - 7x + 3$ as $2x^3 + 0x^2 - 7x + 3$, and perform the multiplication as shown at the right.

$$\begin{array}{r\|r} & \text{TEST} \\ 2 + 0 - 7 + 3 & -2 \\ \underline{2 - 5 } & \underline{-3} \\ 4 + 0 - 14 + 6 & \\ \underline{-10 - 0 + 35 - 15} & \\ 4 - 10 - 14 + 41 - 15 & +6 \end{array}$$

$\therefore\ (2x - 5)(2x^3 - 7x + 3) = 4x^4 - 10x^3 - 14x^2 + 41x - 15.$

WRITTEN EXERCISES

Multiply by detached coefficients and test:

1. $\begin{array}{l} m + n \\ \underline{2m - 3n} \end{array}$ **3.** $\begin{array}{l} a^2 + 2a + 1 \\ \underline{a + 1 } \end{array}$ **5.** $\begin{array}{l} y^3 - y^2 + 5 \\ \underline{2y + 3 } \end{array}$

2. $\begin{array}{l} x^3 - 5x + 3 \\ \underline{x - 1 } \end{array}$ **4.** $\begin{array}{l} x^4 - 3x^3 + 5 \\ \underline{x + 2 } \end{array}$ **6.** $\begin{array}{l} x^4 + 2x^2 + 7x \\ \underline{2x - 1 } \end{array}$

CHAPTER VIII

TYPE PRODUCTS. FACTORS AND MULTIPLES

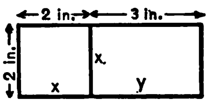
TYPE PRODUCTS

114. Certain products are specially important because they serve as types or models for other multiplications. They apply to positive and negative numbers alike.

115. PREPARATORY.

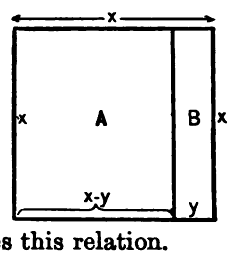
1. Draw a rectangle 4 in. long and 3 in. wide; what is its area? Call the rectangle a in. long and b in. wide; what is its area? Call the length of the rectangle x and the width y; what is its area?

2. Draw a rectangle like the one here shown, having the dimensions indicated outside. What is the area of the square part? Of the other part?



3. Replace the 2 in. by x and the 3 in. by y; what is the area of the square? Of the other part? Write an expression for the sum of these areas.

4. Using x and y, what expression represents the base of the above rectangle? The altitude? The area of the whole figure? How must this result compare with the result of Exercise 3? Write the equation that expresses this fact.



5. Write the expression for the area of this square of side x. For the area of rectangle B. For the difference between these areas.

6. Write the expression for the area of the rectangle A. How does this part compare with the area of the whole figure less B? Write the equation that expresses this relation.

116. Type I: $x(y+z)=xy+xz.$

Type II: $x(y-z)=xy-xz.$

For example:

$a(b+c)=ab+ac.$

$5x(3-y)=15x-5xy.$

$2a^2(a-5b)=2a^3-10a^2b.$

$-3ab(c^2-4ad+b)=-3abc^2+12a^2bd-3ab^2.$

WRITTEN EXERCISES

Multiply:

1. $-x(x+y).$
2. $c(a-b).$
3. $a(t+t^2).$
4. $cx(w+z).$
5. $-y(x-y).$
6. $t(u+\frac{1}{2}at).$
7. $4ab(a+2b).$
8. $5xy(x^2-y^2).$
9. $pq(m-n).$
10. $-2x(3x^2-2xy).$
11. $-3a^xb^yx(a^2-b^2).$
12. $\frac{2}{3}xy(\frac{1}{2}x^2y^2-1).$
13. $-4x^2(3x-2y).$
14. $2m^2(m-n^2).$
15. $3y(4x-y).$
16. $(5a^2-4b^2)(-a^2b^2).$
17. $(5x-acy)(-acxy).$
18. $(3a^2x-8ax^3)(-3a^2x).$
19. $(6am^p+2bn^q)(-6m^2n).$
20. $(9a^3b^2-3cd^2)(-abcd).$
21. $x(y+z+w).$
22. $-3ab(a^2-b^2+c^2).$
23. $\pi(r_1^2+r_2^2+r_1r_2).$

117. Factoring by Types I and II. *A factor of every term of a polynomial is a factor of the polynomial.*

For example:

The factors of $ab+ac$ are a and $b+c$, because $a(b+c)=ab+ac.$

The factors of $3x^2y-6xy^2-12x^2y^2$ are $3xy$ and $x-2y-4xy$ because $3xy(x-2y-4xy)=3x^2y-6xy^2-12x^2y^2.$

ORAL EXERCISES

Read the numbers to fill the blanks:

1. $(a^2)b-(a^2)c=(\quad\quad)(b-c).$
2. $(2x)y-(2x)z=(\quad\quad)(y-z).$
3. $(2a^2)b^2-(2a^2)bc=2a^2(\quad\quad).$

4. $(2\,xy)\,y - (2\,xy)\,z = 2\,xy\,(\quad)$.
5. $(5\,a)\,b - (5\,a)\,c = (\quad)(b - c)$.
6. $7 \cdot 58 + 7 \cdot 29 - 7 \cdot 87 = 7\,(\quad)$.
7. $(ab)\,x + (ab)\,2\,y = (\quad)(x + 2\,y)$.
8. $7 \cdot 5\,a - 7 \cdot 3\,b - 7 \cdot 2\,c = 7\,(\quad)$.

WRITTEN EXERCISES

Write each of the following as a product of two factors:

1. $ax - bx$.
2. $bn - nq$.
3. $x + x^2$.
4. $ab - b$.
5. $r^2 - 3\,r$.
6. $x^2 - x^{4+p}$.
7. $14\,a - 21\,d$.
8. $12\,mp^x - 16\,p^2r$.
9. $32\,ad + 20\,ac$.
10. $48\,q + 6\,r$.
11. $10\,a + 25\,b$.
12. $x^3 - 5\,x^{n+3}$.
13. $16\,a^2 - 12\,a^3$.
14. $15\,x - 9\,y$.
15. $27\,z - 30\,t$.
16. $32\,ax - 48\,cx$.
17. $18\,bc + 45\,ab$.
18. $28\,mq - 49\,pq$.
19. $ax + hx + cx$.
20. $2\,ac + 8\,bc - 12\,c^2$.
21. $10\,xy + 15\,x^2 - 20\,xz$.
22. $64 - 3 \cdot 64 + 8 \cdot 64$.
23. $ab - ac + da - axy - 2\,ba$.
24. $3\,ap + 6\,bq - 21\,cq + 12\,aq$.

118. PREPARATORY.

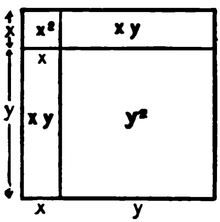


FIG. 1.

1. Show by multiplication that $x + y$ multiplied by $x + y$ equals $x^2 + 2\,xy + y^2$.

2. Show from Fig. 1 that the rectangle of sides $x + y$ and $x + y$, or the square of side $x + y$, is the square on x plus the square on y plus twice the rectangle of x and y.

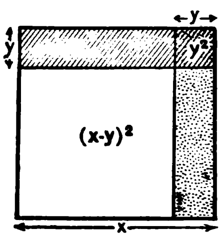


FIG. 2.

3. Find $(x - y)^2$ by multiplication.

4. Find $(x - y)^2$ by use of Fig. 2.

Note that the whole figure is x^2; the entire part shaded with lines is xy; the entire part dotted is xy; while the small square both shaded and dotted is y^2.

119. Type III: $(x+y)^2 = x^2 + 2xy + y^2.$

In words:

The square of the sum of two numbers is the square of the first, plus twice the product of the first and second, plus the square of the second.

120. Type IV: $(x-y)^2 = x^2 - 2xy + y^2.$

In words:

The square of the difference of two numbers is the square of the first, minus twice the product of the first and the second, plus the square of the second.

For example:

$$(a+b)^2 = a^2 + 2ab + b^2.$$
$$(2a-b)^2 = (2a)^2 - 2(2a)b + b^2$$
$$= 4a^2 - 4ab + b^2.$$
$$(s^2-t^2)^2 = (s^2)^2 - 2s^2t^2 + (t^2)^2$$
$$= s^4 - 2s^2t^2 + t^4.$$
$$23^2 = (20+3)^2 = 20^2 + 2 \cdot 20 \cdot 3 + 3^2$$
$$= 400 + 120 + 9 = 529.$$

WRITTEN EXERCISES

Square as indicated:

1. $(n+u)^2$.
2. $(x^2+y^2)^2$.
3. $(a+3b)^2$.
4. $(m+2n)^2$.
5. $(3x+2y)^2$.
6. $(a+2)^2$.
7. $(a^2+1)^2$.
8. $(a-b)^2$.
9. $(x-1)^2$.
10. $(2x-1)^2$.
11. $(x-\frac{1}{2})^2$.
12. $(t+w)^2$.
13. $(x+1)^2$.
14. $(2x+1)^2$.
15. $(2x^2+1)^2$
16. $(2x^2+3y^2)^2$.
17. 25^2 or $(20+5)^2$.
18. 41^2.
19. 82^2.
20. 76^2 or $(80-4)^2$.
21. $(a^2-b^2)^2$.
22. $(3a-2b)^2$.
23. $(t-u)^2$.
24. $(abc-1)^2$.
25. $(mn+u^2)^2$.
26. $(2ab+bc)^2$.
27. $(ahc+1)^2$.
28. $(4x+1)^2$.
29. 33^2.
30. 52^2.
31. 91^2.
32. 17^2.
33. $(5+2x)^2$.
34. $(mb-bc)^2$.
35. 97^2.
36. 46^2.
37. 89^2.
38. 11^2.
39. 36^2.

121. Trinomials may also be squared by Types III and IV.

For example:

$$(a+b+c)^2 = (\overline{a+b}+c)^2 = (a+b)^2 + 2(a+b)c + c^2$$
$$= a^2 + 2ab + b^2 + 2ca + 2bc + c^2$$
$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$(2x - y + z)^2 = (2x - y)^2 + 2(2x - y)z + z^2$$
$$= 4x^2 - 4xy + y^2 + 4xz - 2yz + z^2$$

In words:

The square of a polynomial is the sum of the squares of each of its terms and twice the product of every two.

WRITTEN EXERCISES

Square the trinomials as indicated:

1. $(\overline{a+b}-c)^2$.
2. $(2a-\overline{b+c})^2$.
3. $(a-\overline{3b-c})^2$.
4. $(x-y+z)^2$.
5. $(x+w-2z^3)^2$.
6. $(\frac{1}{2}a-\frac{1}{3}b+\frac{1}{5}c)^2$.
7. $(mn+pq+rs)^2$.
8. $(1-6y+y^2)^2$.
9. $(m^2+mp-q^2)^2$.

122. Square Root. A **square root** of a given number is a number whose second power (or square) equals the given number.

Thus, 7 is a square root of 49, f is a square root of f^2; ab is a square root of a^2b^2; $a+b$ is a square root of $(a+b)^2$; and $ab^2(b+c)$ is a square root of $a^2b^4(b+c)^2$.

We say "a square root," not "the square root," because, as we shall see (Sec. 125), each of these numbers has another square root.

123. Evolution. Finding a root of a number is called **evolution.**

ORAL EXERCISES

1. Name a square root of 9. Of 64. Of 16.

2. If the side of a square is s, what is its area? If the area of a square is s^2, what is its side?

3. What is a square root of a^2? Of s^2? Of a^2b^2? Of x^2y^2?

State a square root of:

4. $4r^2$.
5. $25a^2b^2$.
6. $16a^2b^2c^2$.
7. $4a^4b^4c^4$.
8. $36(x+y)^2$.
9. $a^2(b+c)^2$.
10. $9m^2n^2$.
11. $9(m+n)^2$.
12. $9n^2(a+b)^2$.

124. PREPARATORY.

1. $(+2)(+2)=?$ $(-2)(-2)=?$

2. State a number which taken twice as a factor produces 4. State another number which taken twice as a factor produces 4.

3. According to Exercise 2, how many square roots has 4? What are they?

4. Similarly, name the square roots of 9; 16; 25; 36.

125. Signs of Square Roots. Every number has two square roots which differ only in their signs.

Thus, $\sqrt{4}=+2$ or -2; because $(+2)(+2)=4$, and
$(-2)(-2)=4$.
$\sqrt{a^2}=+a$ or $-a$; because $(+a)(+a)=a^2$, and
$(-a)(-a)=a^2$.

It should be noticed that, although either square root taken twice as a factor produces the given number, the product of the two square roots is not equal to the given number.

126. The sign $\pm$ is used to denote that a number may be taken either positively or negatively.

Thus, $\sqrt{4}=+2$ or -2 is written $\sqrt{4}=\pm 2$.
Also, $\sqrt{a^2}=+a$ or $-a$ is written $\sqrt{a^2}=\pm a$.

ORAL EXERCISES

State the two square roots of each number:

1. 25.	5. 225.	9. 144.	13. $49\,a^2x^4$.
2. 49.	6. 81.	10. 36.	14. $25\,p^2q^6$.
3. 121.	7. 625.	11. a^2b^2.	15. $\frac{1}{4}\,g^2t^2$.
4. 196.	8. 169.	12. $36\,a^2b^2$.	16. $\frac{9}{16}\,m^2n^8$.

127. The square roots of a monomial may often be found by factoring.

EXAMPLE

Find the square root of $576\,m^2n^4$.

By trial, $576=2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot3\cdot3=2^3\cdot2^3\cdot3\cdot3$,
and $m^2n^4=mn^2\cdot mn^2$.

$$\therefore \sqrt{576\,m^2n^4}=\pm 2^3\cdot3\cdot m\cdot n^2$$
$$=\pm 24\,mn^2.$$

WRITTEN EXERCISES

Find by factoring:

1. $\sqrt{625\ m^2n^2}$.
2. $\sqrt{11025\ x^6}$.
3. $\sqrt{256\ a^4b^4}$.
4. $\sqrt{225\ x^6y^6}$.
5. $\sqrt{3025\ x^2y^2}$.
6. $\sqrt{961\ a^6b^8}$.
7. $\sqrt{3136\ m^4n^8}$.
8. $\sqrt{169(a+b)^2}$.
9. $\sqrt{9216(x-y)^4}$.

128. Factoring by Types III and IV. The equal factors of any trinomial that is the square of a binomial may be found by reference either to $(x+y)^2 = x^2 + 2xy + y^2$, or to $(x-y)^2 = x^2 - 2xy + y^2$.

For example:

The factors of $m^2 + 2mr + r^2$ are $m+r$, $m+r$, because

$$(m+r)^2 = (m+r)(m+r) = m^2 + 2mr + r^2.$$

The factors of $4a^2 - 4a + 1$ are $2a-1$, $2a-1$, because

$$(2a-1)^2 = (2a-1)(2a-1) = 4a^2 - 4a + 1.$$

129. A trinomial is the square of a binomial, if one term is twice the product of the square roots of the other two, and not otherwise. The signs of the two square roots must be taken so that their product shall have the sign of the given third term.

ORAL EXERCISES

Read and supply the blanks:

1. $x^2 + 2xy + y^2 = (x+y)(\qquad)$.
2. $4x^2 + 4xy + y^2 = (2x+y)(\qquad)$.
3. $1 + 4m + 4m^2 = (\qquad)(1+2m)$.
4. $m^2n^2 - 2mn + 1 = (mn-1)(\qquad)$.
5. $9x^2 - 24xy + 16y^2 = (3x-4y)(\qquad)$.
6. $25a^2 + 50ab + 25b^2 = (5a+5b)(\qquad)$.

WRITTEN EXERCISES

Write the equal factors of each of the following:

1. $c^2 - 2cd + d^2$.
2. $t^2 - 2tu + u^2$.
3. $4x^2 + 4x + 1$.
4. $a^2x^2 - 2ax + 1$.

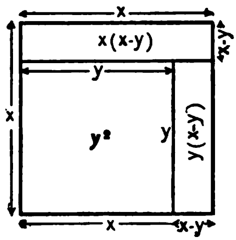
5. $25 + 20x + 4x^2$.
6. $m^2n^2 - 2mn^3 + n^4$.
7. $4a^2b^2 + 4ab^2c + b^2c^2$.
8. $16x^2 - 8x + 1$.
9. $a^2b^2c^2 + 2abc + 1$.
10. $4x^4 - 12x^2y^2 + 9y^4$.
11. $a^4 - 2a^2b^2 + b^4$.
12. $9a^2 + 12a + 4$.

Express each of the following as the square of a binomial, or prove that it cannot be so expressed:

13. $t^2 - 2t + 2$.
14. $a^4 - 3a^2 + 4$.
15. $z^2 - \frac{2}{3}xz + \frac{1}{9}x^2$.
16. $1 - 4x + 4x^2$.
17. $9r^2 - 18r + 10$.
18. $a^2b^2 - 8ab + 20$.

130. Preparatory.

1. Find $(x + y)(x - y)$ by multiplication.



2. According to the figure the whole area x^2 when diminished by y^2 leaves what two rectangles?

3. If these rectangles are placed end to end, a rectangle is formed whose width is $x - y$, and whose length is $x + y$. From Exercise 2, what does the rectangle $(x - y)(x + y)$ equal?

131. Type V: $(x + y)(x - y) = x^2 - y^2$.

In words:

The product of the sum and the difference of two numbers is the difference of their squares.

For example:

$$(a + b)(a - b) = a^2 - b^2.$$
$$(2a + b)(2a - b) = (2a)^2 - b^2 = 4a^2 - b^2.$$
$$(m^2 + n^2)(m^2 - n^2) = (m^2)^2 - (n^2)^2 = m^4 - n^4.$$

ORAL EXERCISES

Multiply:

1. $(m-n)(m+n)$.
2. $(a-x)(a+x)$.
3. $(p-q)(p+q)$.
4. $(t+u)(t-u)$.
5. $(x-1)(x+1)$.
6. $(x-2)(x+2)$.
7. $(x+b)(x-b)$.
8. $(2x-1)(2x+1)$.
9. $(2x-y)(2x+y)$.

WRITTEN EXERCISES

Multiply:

1. $(1+x^3)(1-x^3)$.
2. $(ax+by)(ax-by)$.
3. $(2x-3y)(2x+3y)$.
4. $(2a^2+3)(2a^2-3)$.
5. $(a^2+b)(a^2-b)$.
6. $(a^2-3ax)(a^2+3ax)$.
7. $(ax-x^2)(ax+x^2)$.
8. $(\frac{1}{2}x-\frac{1}{3}y)(\frac{1}{2}x+\frac{1}{3}y)$.
9. $(a-x)(a+x)(a^2+x^2)$.
10. $(a-x)(a+x)(a^2+x^2)(a^4+x^4)$.
11. $(1-r)(1+r)(1+r^2)(1+r^4)(1+r^8)$.
12. $(1-r)(1+r)(1+r^2)(1+r^4)(1+r^8)(1+r^{16})$.

132. Two numbers, one greater than a multiple of 10, and the other less than this multiple by the same amount, may be multiplied as the sum and difference of two numbers, according to Type V.

Thus, $93 \cdot 87 = (90+3)(90-3) = 90^2 - 3^2 = 8100 - 9 = 8091$.

WRITTEN EXERCISES

1. $31 \cdot 29 = (30+1)(30-1) = ?$
2. $42 \cdot 38 = (40+2)(40-2) = ?$
3. $35 \cdot 45 = ?$
4. $57 \cdot 63 = ?$
5. $21 \cdot 19 = ?$
6. $32 \cdot 28 = ?$
7. $29 \cdot 31 = ?$
8. $66 \cdot 54 = ?$
9. $44 \cdot 36 = ?$
10. $91 \cdot 89 = ?$
11. $53 \cdot 47 = ?$
12. $16 \cdot 24 = ?$
13. $99 \cdot 101 = ?$
14. $98 \cdot 102 = ?$
15. $90 \cdot 110 = ?$
16. $127 \cdot 113 = ?$
17. What is the cost of 21 doz. eggs at 19¢ a dozen?
18. What is the cost of 28 lb. of butter at 32¢ a pound?
19. How many oranges in 146 crates of 154 oranges each?
20. What is the area of a rectangle whose dimensions are 62 ft. and 58 ft.?
21. How far does a train travel in 37 hours at the rate of 43 miles per hour?
22. What is the cost of 102 shirt waists at 98¢ each?
23. What is the cost of 88 yd. of carpet at 92¢ a yard?

133. Factoring by Type V. The factors of the difference of two squares are the sum and the difference of the numbers whose squares are given.

For example:

The factors of $x^2 - y^2$ are $x + y$, $x - y$, because

$$(x + y)(x - y) = x^2 - y^2.$$

The factors of $a^2b^2 - 1$ are $ab + 1$, $ab - 1$, because

$$(ab + 1)(ab - 1) = a^2b^2 - 1.$$

The factors of $a^4 - 4c^2d^2$ are $a^2 + 2cd$, $a^2 - 2cd$, because

$$(a^2 + 2cd)(a^2 - 2cd) = a^4 - 4c^2d^2.$$

ORAL EXERCISES

Read and supply the blanks:

1. $t^2 - v^2 = (t - v)(\quad)$.
2. $a^{2x} - 4b^{2y} = (a^x - 2b^y)(\quad)$.
3. $4x^2 - y^2 = (2x + y)(\quad)$.
4. $9x^2 - 4y^2 = (3x - 2y)(\quad)$.
5. $a^2x^{2m} - y^{2n} = [ax^m - (\ \ )][ax^m + (\ \ )]$.
6. $25s^2 - 49t^2 = [5s + (\ \ )][5s - (\ \ )]$.

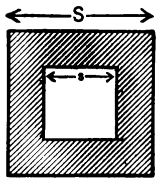
WRITTEN EXERCISES

Write the factors of:

1. $1 - y^2$.	7. $4x^4 - y^4$.	13. $26^2 - 24^2$.
2. $81u^2 - 64v^2$.	8. $a^4 - 9b^4$.	14. $41^2 - 31^2$.
3. $121t^2 - 4$.	9. $x^{10} - y^8$.	15. $32^2 - 28^2$.
4. $1 - 144q^2$.	10. $y^4 - \frac{1}{4}$.	16. $a^2 - 36b^4$.
5. $x^2y^2 - 25$.	11. $92^4 - 1$.	17. $763^2 - 663^2$.
6. $144a^2b^2 - 49c^2$.	12. $36x^4 - 49y^4$.	18. $36x^2y^2 - 169t^2u^2$.

19. Calculate the area of the shaded portion of this square, if

	(1)	(2)	(3)	(4)	(5)	(6)
$S =$	49	290	597	73	$6a$	$24q$
$s =$	45	280	497	27	$4a$	$14q$



20. A walk 8 ft. wide is laid around the inside of the fence of a square lot 92 ft. square. Find the area of the walk.

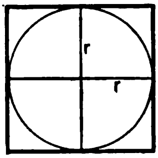


FIG. 1.

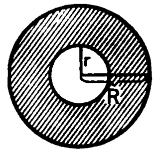


FIG. 2.

21. It is seen from Figure 1 that the area inclosed by a circle is less than 4 times the square of the radius; it is very nearly $\frac{22}{7}$ or 3.1416 times the square of the radius. The exact multiple is commonly denoted by π and the area of a circle by πr^2. Using $\frac{22}{7}$ for π, find the area of the shaded part of Figure 2, if

	(1)	(2)	(3)	(4)	(5)
$R=$	28	301	2743	$4x$	$53d$
$r=$	21	289	2643	$2x$	$49d$

134. Type VI: $(x+a)(x+b)=x^2+(a+b)x+ab$.

For example:

$(x+5)(x+3)=x^2+8x+15.$

$(x-3)(x+7)=x^2+4x-21.$

$(a+4)(a-6)=a^2+(4-6)a+4(-6)=a^2-2a-24.$

$(a+x)(a+y)=a^2+(x+y)a+xy.$

$(3x+c)(3x-d)=(3x)^2+(c-d)3x-cd=9x^2+3(c-d)x-cd.$

$$91\cdot 87=(100-9)(100-13)=100^2-22\cdot 100+9\cdot 13$$
$$=10000-2200+117=7917.$$

ORAL EXERCISES

State the products:

1. $(x+2)(x+5)$.
2. $(a+3)(a+6)$.
3. $(b-5)(b+2)$.
4. $(4x+7)(4x-5)$.
5. $(5+m)(5+r)$.
6. $(4g-5)(4g-9)$.
7. $(t-2)(t-3)$.
8. $(y+1)(y-5)$.
9. $(p+q)(p+2q)$.
10. $(7-x)(7-y)$.
11. $(ab+c)(ab+d)$.
12. $(2a-7b)(2a+8b)$.

WRITTEN EXERCISES

Find the products:

1. $93 \cdot 95$.
2. $197 \cdot 191$.
3. $(x+14)(x-19)$.
4. $(2z-3a)(2z+5a)$.
5. $993 \cdot 985$.
6. $(-3x+11)(-3x-6)$.
7. $(15x+23)(15x-25)$.
8. $(4w+a)(4w-2a)$.

135. Type VII. $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

Since by actual multiplication

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

therefore, $(x+a)^3 = (\ )^3 + 3(\ )^2a + 3(\ )a^2 + (\ )^3$.

Also, $(a+b^2)^3 = (\ )^3 + 3(\ )b^2 + 3(\ )(b^2)^2 + (b^2)^3$
$= (\ ) + 3(\ ) + 3(\ ) + (\ )$.

Since by actual multiplication

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3,$$

therefore, $(m-n)^3 = (\ )^3 - 3(\ ) + 3(\ ) - (\ )^3$.

Similarly, $(ab-c)^3 = (ab)^3 - 3(ab)^2c + 3abc^2 - c^3$
$= (\ ) - 3(\ ) + 3abc^2 - c^3$.

Similarly, $(\overline{a-b}+c)^3 = (a-b)^3 + 3(a-b)^2c + 3(a-b)c^2 + c^3$
$= (\ ) + 3(\ ) + 3(\ ) + c^3$.

WRITTEN EXERCISES

Expand by Type VII:

1. $(m+n)^3$.
2. $(p-q)^3$.
3. $(a-x)^3$.
4. $(2a+x)^3$.
5. $(a^2+b^2)^3$.
6. $(m^2-n^2)^3$.
7. $(a+2b)^3$.
8. $(a-3c)^3$.
9. $(2a-3b)^3$.
10. $a(a+b)^3$.
11. $ax(x-y)^3$.
12. $(ab+cd)^3$.
13. $(x+1)^3$.
14. $(3x-1)^3$.
15. $(y^2-1)^3$.
16. $(a^m+b^m)^3$.
17. $(x^n-y^n)^3$.
18. $(\frac{1}{2}x-\frac{1}{3}y)^3$.
19. $(x^n-y^m)^3$.
20. $(a^2b+ab^2)^3$.
21. $(\overline{m+n}-p)^3$.
22. $(m+\overline{n-p})^3$.
23. $(a+b+c)^3$.
24. $(2a+b+c)^3$.

136. Cube Root. **A cube root** of a given number is a number whose third power (or cube) equals the given number.

For example:

4 is a cube root of 64 because $4 \cdot 4 \cdot 4 = 64$.

a is a cube root of a^3 because $a \cdot a \cdot a = a^3$.

bc is a cube root of b^3c^3 because $bc \cdot bc \cdot bc = b^3c^3$.

$3ab$ is a cube root of $27a^3b^3$.

$2(a+b)$ is a cube root of $8(a+b)^3$.

$a(b+c)$ is a cube root of $a^3(b+c)^3$.

ORAL EXERCISES

Name a cube root of:

1. a^6.
2. $27a^3$.
3. $8a^3b^3$.
4. $8x^3y^3z^3$.
5. $27x^3y^3m^6$.
6. $8(x+y)^3$.
7. $a^3(b+c)^3$.
8. $8m^3(m+n)^3$.
9. $125(p-q)^3$.
10. $x^6y^6(b-c)^3$.
11. $(a+b)^3(x+y)^3$.
12. $(x-y)^6(x+y)^3$.

137. Square root and cube root are used to solve equations.

EXAMPLES

1. Solve: $x^2 = 16$. (*1*)

Extracting the square root of each side, $x = \pm 4$. (*2*)

2. Solve: $x^2 - 64 = 0$. (*1*)

Adding 64 to each member of (*1*), $x^2 = 64$. (*2*)

Extracting the square root of each side of (*2*), $x = \pm 8$. (*3*)

3. Solve: $x^2 - 5 = 0$. (*1*)

Adding 5 to both members of (*1*), $x^2 = 5$. (*2*)

Extracting the square root of each side of (*2*), $x = \pm\sqrt{5}$. (*3*)

4. A cubical block of stone contains 1728 cu. in. What is the length of an edge?

Solution.

1. Let e be the number of inches in the length of an edge; then, $e^3 = 1728$.

2. Extracting the cube root, $e = 12$.

3. $\therefore$ the block is 12 in. long.

Test. $12^3 = 1728$.

5. A building has 3 square floors, with a total area of 4800 sq. ft. What is the width of each floor?

SOLUTION.

Let	x = the number of ft. in the width.	(1)
Then,	x^2 is the number of sq. ft. in the area of one floor, and	(2)
	$3x^2$ is the number of sq. ft. in the area of the three floors.	
Then,	$3x^2 = 4800$, by the given conditions.	(3)
Dividing by 3,	$x^2 = 1600$.	
$\therefore$	$x = 40$.	(4)

Therefore, the width of each floor was 40 ft.

TEST. $x^2 = 1600$ and $3x^2 = 4800$.

$40 \cdot 40 \cdot 1$ sq. ft. $= 1600$ sq. ft. $3 \cdot 1600$ sq. ft. $= 4800$ sq. ft.

WRITTEN EXERCISES

Solve each of the following equations:

1. $x^2 = 144$.	7. $q^2 - 81 = 0$.	13. $s^2 - \frac{1}{4} = 0$.
2. $x^2 = 81$.	8. $r^2 - 144 = 0$.	14. $t^2 - \frac{9}{16} = 0$.
3. $x^2 = 49$.	9. $z^2 - 625 = 0$.	15. $r^2 - .25 = 0$.
4. $d^2 - 25 = 0$.	10. $u^2 - 100 = 0$.	16. $w^2 - \frac{16}{25} = 0$.
5. $x^2 - 121 = 0$.	11. $3v^2 - 75 = 0$.	17. $2s^2 - \frac{8}{9} = 0$.
6. $x^2 - 169 = 0$.	12. $5w^2 - 500 = 0$.	18. $3x^2 - \frac{12}{25} = 0$.

19. The following is an important formula of physics, in connection with the steam engine:

$$A = \frac{\pi D^2}{4} - \frac{\pi d^2}{4}$$

Substituting the numbers of the following table in the above equation, and using $\frac{22}{7}$ as the value of π, find the numbers to fill the blanks:

	(1)	(2)	(3)	(4)
$A =$	$\frac{561}{14}$	$\frac{440}{7}$	——	$\frac{517}{2}$
$D =$	——	12	24	——
$d =$	7	——	15	20

20. In the equation $16t^2 = 256$, t is the number of seconds required for a body to fall 256 ft. How many seconds is this?

21. A cubical pier was made smaller by cutting off 2000 cu. ft.; its volume was then 6000 cu. ft. Find the length of an edge of the original cube.

22. The area inclosed by a circle is πr^2 (p. 86). Using 3.1416 as the approximate value of π, find the radius of a circle whose area is 12.5664 sq. in.

23. Find the radius of a circle whose area is 3.1416 sq. ft.

24. Find the diameter of a circle whose area is 28.2744 sq. yd.

FACTORS AND MULTIPLES

138. Integral Expressions. An algebraic expression is called **integral** if it has not the *form* of a fraction.

Thus, $3a$ is an integral expression, but $\frac{a}{2}$ is not an integral expression.

Expressions are sometimes spoken of as *integral with respect to a letter*. For example, $\frac{a}{2}$ is integral with respect to a, although it is a fraction.

In what follows, the terms "factor" and "divisor" are to be understood to mean "integral factor" and "integral divisor."

139. Common Factor. An expression that is a factor of each of two or more expressions is called a **common factor** of the expressions.

140. Highest Common Factor. The highest common factor (h. c. f.) of two or more expressions is the product of their literal common factor of highest degree, and the greatest common divisor of their numerical coefficients, taken positively.

For example, to find the h. c. f. of $3a^2b$, $-6ab^2$, and $9abc$:

The literal common factor of highest degree is ab. The greatest common divisor (g. c. d.) of 3, -6, and 9 is 3. Hence the h. c. f. of $3a^2b$, $-6ab^2$, and $9abc$ is $3ab$.

Although -3 is also a common divisor of 3, -6, and 9, it is customary to take the g. c. d. with the positive sign.

If the given expressions are factored so as to have the h. c. f. as one factor, the set of second factors will have no further common factor, other than unity.

141. *In the case of monomials,* the h. c. f. is seen by inspection. Its coefficient is the g. c. d. of the given numerical coefficients, and its literal part consists of each letter with the lowest exponent which it has in any of the monomials.

If expressions not monomials are given, they must first be factored if possible, after which the h. c. f. can usually be seen.

Thus, find the highest common factor of $ab^2 + abc$, and $b^2c + bc^2$: Factoring,

$$ab + abc = ab(b+c).$$

$$b^2c + bc^2 = bc(b + c).$$

The h. c. f. is the product of the common factors b and $b + c$, or $b(b + c)$.

WRITTEN EXERCISES

Find the h. c. f. of:

1. $3xy^3,\ x^2y.$
2. $x^4y^3,\ x^3y,\ xy^3.$
3. $10a^2,\ 15a^3,\ 5a.$
4. $x^2 + xy,\ (x + y)^3.$
5. $3x^4,\ 2x^3,\ 4x^5,\ x^3.$
6. $3a^2bc^3,\ 15ab^3c^2,\ 10a^2b^2.$
7. $4a^3b^2c,\ 8ab^5c^2,\ 12xybc^2.$
8. $10a^2b^4c,\ 15ab^3c^2,\ 20a^2c^3.$
9. $3ax^2,\ -2a^2x,\ a^2x^2,\ -3abx.$
10. $3a^3 + 2a^2b - 5ab^2,\ 2a^2b + 2ab^2.$

142. Multiples. A product is called a **multiple** of any of its factors.

Thus, abc is a multiple of a, of b, of c, of ab, of bc, and of ac.

Also, x^4 is a multiple of x, x^2, x^3, and x^4.

143. Common Multiple. An expression that is a multiple of two or more expressions is called a **common multiple** of these expressions.

Thus, $12x^3y^2$ is a common multiple of $3xy$ and $6x^3$.

144. Lowest Common Multiple. The lowest common multiple (l. c. m.) of two or more expressions is the product of their literal common multiple of lowest degree, and the least common multiple of their numerical coefficients, taken positively.

For example, find the l. c. m. of $8a^2b$, $6abc$, and $-4ac^3$:

The literal common multiple of lowest degree is a^2bc^3. The least common multiple of 8, 6, and -4 is 24.

Thus, the lowest common multiple of $8a^2b$, $6abc$, and $-4ac^3$ is $24a^2bc^3$.

Although -24 is a common multiple of 8, 6, and -4, it is customary to take the lowest common multiple with the positive sign.

If the least common multiple is divided by each of the given expressions in turn, the quotients will have no common divisor, other than unity.

145. *In the case of monomials*, the lowest common multiple is seen by inspection. Its numerical coefficient is the least common multiple of the given coefficients, and its literal part consists of the various letters occurring in any of the monomials, each letter with the highest exponent which it has in any of the given expressions.

If expressions not monomials are given, they must first be factored if possible, after which the factors of the lowest common multiple may be seen.

1. Find the l.c.m. of ax, $ac + ab$, and $acx^2 + abx^2$.

   1. $ax = ax$.
   2. $ac + ab = a(b + c)$.
   3. $acx^2 + abx^2 = ax^2(b + c)$.
   4. $\therefore$ the l. c. m. is the product of a, x^2, and $b + c$, or $ax^2(b + c)$.

WRITTEN EXERCISES

Find the l. c. m. of:

1. $4ab$, $6ac$.
2. $5a^2$, $10ax$.
3. $6pr$, $9pq$.
4. $7x^2$, $3xy$.
5. xyz, yzw.
6. abc^2, a^2b^2c.
7. $x + y$, $ax + ay$.
8. $a^2 + ac$, $ab + bc$.
9. $bcx + bcy$, abc.
10. $13a^2 - 13b^2$, $39ab$.
11. $ax + xy$, $abc + bcy$.
12. pq, $apq - bpq$.
13. $a(b - c)$, $xb - xc$.
14. $17x^2$, $51y^2$, $17a$.
15. $(a + b)n$, $(a + b)r$.
16. $3a^2bc$, $5a^3b^2$, $15a^2b^3c$.
17. $(t - u)x$, $(t - u)xyz$.
18. $8xyz^2$, $24x^2y^2z$, $6xy^2z^2$.

SUMMARY

I. Definitions.

1. An *integral expression* is one not in the form of a fraction. Sec. 138.

2. A *common factor* of two or more expressions is a factor of each of them. Sec. 139.

3. The *highest common factor* of two or more expressions is the product of their literal common factor of highest degree, and the g. c. d. of their numerical coefficients, taken positively. Sec. 140.

4. A product is called a *multiple* of any of its factors. Sec. 142.

5. A *common multiple* of two or more expressions is a multiple of each of them. Sec. 143.

6. The *lowest common multiple* of two or more expressions is the product of their literal common multiple of lowest degree, and the l. c. m. of their numerical coefficients, taken positively. Sec. 144.

7. A *square root* of a given number is a number whose second power (or square) equals the given number. Sec. 122.

8. Every number has two square roots which differ only in their signs.

9. A *cube root* of a given number is a number whose third power (or cube) equals the given number. Sec. 136.

10. *Evolution* is the process of finding a root of a number. Sec. 123.

II. Processes.

1. In the case of monomials the h. c. f. is seen by inspection. Its coefficient is the greatest common divisor of the given numerical coefficients, and its literal part consists of each letter with the lowest exponent which it has in any of the monomials. Sec. 141.

If expressions not monomials are given, they must first be factored if possible, after which the h. c. f. can usually be seen. Sec. 141.

2. In the case of monomials, the l. c. m. is seen by inspection. Its numerical coefficient is the least common multiple

of the given coefficients, and its literal part consists of the various letters occurring in any of the monomials, each letter with the highest exponent which it has in any of the given expressions.

If expressions not monomials are given, they must first be factored if possible, after which the factors of the lowest common multiple may be seen. Sec. 145.

III. **Important Type Products.**

1. $x(y+z)=xy+xz.$
2. $x(y-z)=xy-xz.$
3. $(x+y)^2=x^2+2xy+y^2.$
4. $(x-y)^2=x^2-2xy+y^2.$
5. $(x+y)(x-y)=x^2-y^2.$
6. $(x+a)(x+b)=x^2+(a+b)x+ab.$
7. $(x+y)^3=x^3+3x^2y+3xy^2+y^3.$

REVIEW

ORAL EXERCISES

1. Express in words each equation of the list in III above.

Taking $3ax$ as one factor of each of the following expressions, state the other factor:

2. $12a^2x.$ 3. $-15ax^3.$ 4. $2ax.$ 5. $-30a^2x^2y.$ 6. $ab^2x.$

State as a product of two factors:

7. $x^2+3x.$
8. $ax+bx.$
9. $4r^2-8r.$
10. $9ab-12ac.$
11. $ab-bc.$
12. $mp-p^2.$

State the square as indicated:

13. $(a+x)^2.$
14. $(x+3)^2.$
15. $(1-x)^2.$
16. $(2a-1)^2.$
17. $(10+3)^2.$
18. $(7-y)^2.$

State the square roots of:

19. $a^2+2at+t^2.$
20. $x^2-2x+1.$
21. $9x^2+6x+1.$
22. $x^6.$
23. $a^2y^4.$
24. $y^2-10y+25.$
25. $16q^8.$
26. $(a+b)^2x^6.$
27. $49x^2y^2z^4.$

State the products:

28. $(a-y)(a+y).$
29. $(3+t)(3-t).$
30. $(20+1)(20-1).$
31. $(3a+x)(3a-x).$

State as a product of two factors:

32. $x^2 - 49$. **33.** $4a^2 - y^2$. **34.** $400 - 25$. **35.** $p^2 - 1$.

Name a cube root of:

36. $8x^3$. **37.** $27y^6$. **38.** $(x-3)^3$. **39.** $64x^6(a-x)^3$.

WRITTEN EXERCISES

Write as a product of two factors:

1. $16x^2 - 48x + 36$.

2. $4a^2x - 8axy + 12a^2xy$.

3. $a^2b^2 - 64c^2d^2$.

4. $5z^2 - 4z$.

5. $25x^2 + 70ax + 49a^2$.

6. $5a^2t^2 - 30a^2ty + 45a^2y^2$.

Find, by factoring, the two square roots of:

7. 7056. **8.** $3136x^2y^4$. **9.** $2025(9 - 6c + c^2)$.

Solve:

10. $x^2 - 256 = 0$.

11. $y^2 = 225$.

12. $z^2 - \frac{16}{49} = 0$.

13. $7x^2 - 63 = 0$.

14. $t^2 - \frac{9}{100} = 0$.

15. $5w^2 - 125 = 0$.

Remove parentheses and unite terms where possible:

16. $(3x-1)^2 + 2(4x+3)^2$.

17. $5(7y-4) - (4y+3)^2$.

18. $(79)^2 + (92)^2$.

19. $(ab-c)^2 + (ab+c)^2$.

20. $(x-3y)(x+3y) + (x-5y)^2$.

21. $3x(7y-4) - (2x+3y)^2$.

22. $133 \cdot 127$ or $(130+3)(130-3)$.

23. $4a(b-1) + 2(3a-b)^2$.

Find the product:

24. $(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})(1+x^{32})$.

25. Show by multiplying that

$$s(s-a)(b+c) + a(s-b)(s-c) - 2bcs$$

is identical with

$$s(s-b)(a+c) - b(a-s)(s-c) - 2acs.$$

26. A circular pipe has an outer diameter of 12 in., and an inner diameter of 8 in. Find the weight of an 8-foot section of the pipe, if one cubic inch of the material of which it is made weighs $1\frac{3}{4}$ oz.

SUPPLEMENTARY WORK

Special Tests

Certain polynomials have properties which aid in testing the work of multiplication.

Thus, in $(x+y)(x^2-xy+y^2)=x^3+y^3$, each term of the first factor is of the first degree, and each term of the second is of the second degree; hence, if each term of the product were not of the third degree, the product would be incorrect.

Expressions all of whose terms are of the same degree are called **homogeneous**; when factors are homogeneous, their product is homogeneous, and its degree is the sum of the degrees of its factors.

ORAL EXERCISES

1. Without multiplying state the degree of each term in the product of $a+b$ and a^2+b^2.

2. Similarly for x^2+xy and $x+y$. Also for $mn+n^2$ and m^2+n^2.

3. Without multiplying determine whether or not x^3+xy+y^3 is the product of x^2+xy+y^2 and $x+y$.

WRITTEN EXERCISES

Multiply and test first by seeing whether the product is homogeneous. If it is homogeneous, test further by substituting arbitrary values:

1. $(a^3+2b^3)(a^3-3b^3)$.
2. $(x+y)(x^2-3xy+5y^2)$.
3. $(x^2-y^2)(x^2+6xy-y^2)$.
4. $(a+x)^2(a^2-x^2)$.
5. $(a^2-5ax+x^2)(x^3+2a^3)$.
6. $(x^2-2xy+y^2)(x^2+y^2)$.

CHAPTER IX

DIVISION

DIVISION OF MONOMIALS

146. Preparatory.

1. 4 lb. $\div 2 = (\quad)$ lb. 4 yd. $\div 2 = (\quad)$ yd. $4y \div 2 = (\quad)y$.
2. 6 oz. $\div 3 = (\quad)$ oz. 6 ft. $\div 3 = (\quad)$ ft. $6f \div 2 = (\quad)f$.
3. $3a \cdot (?) = 6a^2b$, then $6a^2b \div 3a = ?$
4. $8b^2 \cdot (?) = 16ab^3$, then $16ab^3 \div 8b^2 = ?$
5. $a^2b \cdot (?) = a^2bcd$, then $a^2bcd \div a^2b = ?$

147. Division. Division is the process of finding one of two factors when the product and the other factor are given. Division is thus the inverse of multiplication.

The problem $18a^3b^7c \div 3a^2b^4c$ means to find the number by which $3a^2b^4c$ must be multiplied to produce $18a^3b^7c$. To do this: 3 must be multiplied by 6 to produce 18.

a^2 must be multiplied by a to produce a^3.

b^4 must be multiplied by b^3 to produce b^7.

c multiplied by 1 produces c.

Hence, the quotient of $18a^3b^7c \div 3a^2b^4c$ is $6 \cdot a \cdot b^3 \cdot 1$, or $6ab^3$.

148. Law of Exponents. Since $a^5 = a^3 \cdot a^2$, it follows by dividing both members by a^2 that $\frac{a^5}{a^2} = a^3$.

Likewise, from $a^{m+r} = a^m \cdot a^r$, it follows by dividing both members by a^r that $\frac{a^{m+r}}{a^r} = a^m$.

In words:

To divide powers of the same base subtract the exponent of the divisor from that of the dividend.

For example:

$$\frac{a^{12}}{a^7} = a^5, \text{ also } \frac{a^{3n}}{a^n} = a^{2n},$$

$$\frac{a^{n+2}}{a^2} = a^n, \text{ also } \frac{(ab)^5}{(ab)^2} = (ab)^3.$$

149. To find the quotient of two positive monomials *divide the coefficient of the dividend by the coefficient of the divisor, and to this quotient annex each letter with an exponent equal to the exponent of that letter in the dividend, diminished by its exponent in the divisor, omitting any letter having the same exponent in the dividend and divisor.*

150. Test of Division. The product of the divisor and the quotient must equal the dividend.

WRITTEN EXERCISES

Divide and test:

1. $65\,a^2b \div 13\,ab$.
2. $48\,x^2y \div 12\,xy$.
3. $63\,m^2n^2 \div 21\,mn$.
4. $96\,a^3b^2 \div 12\,ab$.
5. $45\,p^2q^3 \div 15\,p^2q^2$.
6. $120\,abc \div 20\,a$.
7. $\frac{1}{2}\,mv^2 \div mv$.
8. $\frac{1}{2}\,gt^2 \div \frac{1}{2}\,t$.
9. $\frac{4}{3}\,\pi r^3 \div \pi r^2$.
10. $\frac{1}{6}\,\pi d^3 \div \frac{1}{6}\,\pi d$.
11. Find the numbers to fill the blanks:

	(1)	(2)	(3)	(4)	(5)
Dividend:	$12\,x^2$	$27\,ab$	$33\,ay$	$42\,p$	$36\,mn$
Divisor:	$3\,x$	——	——	$7\,p$	$9\,m$
Quotient:	——	$3\,b$	$11\,a$	——	——

151. An Important Property of Division. From the properties of the equation (Sec. 20, p. 11) it follows that the equation:

$$\textit{Dividend} = \textit{Divisor} \times \textit{Quotient}$$

remains true if both dividend and divisor are multiplied (or divided) by the same number; for this means multiplying (or dividing) both members of the equation by that number.

In other words:

The quotient is not altered, if both dividend and divisor are multiplied or divided by the same number.

For example:

$$\frac{24}{8} = 3, \text{ because } 3 \cdot 8 = 24,$$

$$\text{and } \frac{2 \cdot 24}{2 \cdot 8} = 3, \text{ because } 2 \cdot 8 \cdot 3 = 2 \cdot 24.$$

$$\text{Similarly, } \frac{24 \div 2}{8 \div 2} = 3, \text{ because } (8 \div 2)\, 3 = 24 \div 2,$$

or, performing the divisions,

$$\frac{12}{4} = 3, \text{ because } 4 \cdot 3 = 12.$$

The division of both dividend and divisor by the same number can be performed without rewriting, by drawing a line through the number divided and writing the results above or below. Unity is not usually written.

Thus:

$$\frac{\overset{12}{\cancel{24}}}{\underset{4}{\cancel{8}}} = \frac{\overset{3}{\cancel{12}}}{\cancel{4}} = 3.$$

This process is called **canceling**. Canceling indicates that dividend and divisor have been divided by the same number.

All that has been said applies as well to numbers represented by letters as to those represented by numerals.

Thus:

$$\frac{\overset{2}{\cancel{8}}\,\cancel{a}\,\overset{b^2}{\cancel{b^3}}}{\cancel{4}\,\cancel{a}\,\cancel{b}} = 2\,b^2.$$

152. Signs of Quotients. The law of signs of quotients follows directly from the law of signs of products. (Sec. 102, p. 65.) If the dividend is positive, its two factors (the divisor and the quotient) must have like signs. If the dividend is negative, the divisor and quotient must have unlike signs.

$$\text{From } \begin{cases} a = bq \\ a = (-b)(-q) \\ -a = b\,(-q) \\ -a = (-b)\,q \end{cases} \text{ it follows that } \begin{cases} \dfrac{a}{b} = q \\ \dfrac{a}{-b} = -q \\ \dfrac{-a}{b} = -q \\ \dfrac{-a}{-b} = q \end{cases}$$

By indicating only the signs of the numbers involved, these relations may be expressed as follows:

$$\frac{+}{+}=+;\quad \frac{+}{-}=-;\quad \frac{-}{+}=-;\quad \text{and}\ \frac{-}{-}=+.$$

In words:

When the *signs* of the dividend and the divisor are *alike*, the quotient has the sign *plus;* when they are *unlike*, the quotient has the sign *minus.*

ORAL EXERCISES

State the quotients:

1. $-6 \div 3$.	6. $18 \div -12$.	11. $ab \div -a$.
2. $-10 \div 4$.	7. $-6 \div -4$.	12. $-ab \div -b$.
3. $-18 \div 12$.	8. $-10 \div -4$.	13. $-a^2b \div -ab$.
4. $16\,a \div -3$.	9. $-18 \div -12$.	14. $-6\,xy \div -x$.
5. $10 \div -4$.	10. $-ab \div a$.	15. $-\frac{1}{2}mv^2 \div \frac{1}{2}v$.

16. $a^5 \div a^3$.
17. $a^7 \div a^5$.
18. $b^{10} \div b^3$.
19. $b^7 \div b^4$.
20. $a^3b \div a^2$.
21. $a^4b^2 \div a^2b$.
22. $m^7 \div m^2$.
23. $m^3n^2 \div mn^2$.
24. $\dfrac{14\,a^3b}{2\,a}$.
25. $\dfrac{-12\,a^{4s}}{10\,a^{3s}}$.
26. $\dfrac{-a^{4m}}{a^{2m}}$.
27. $\dfrac{3\,a^2b^q}{-3\,ab^q}$.
28. $\dfrac{-b^{5n+3}}{b^3}$.
29. $\dfrac{25\,m^{2x}n}{-5\,m^x}$.
30. $\dfrac{-x^3y^{3n}}{xy^n}$.
31. $\dfrac{-p^2qr}{pqr}$.
32. $\dfrac{-65\,x^3y^5}{13\,xy^3}$.
33. $\dfrac{-18\,abx^3y^{2p}}{9\,abxy^p}$.
34. $\dfrac{-6\,a^2b^{x+2}}{3\,ab^2}$
35. $\dfrac{20\,m^2n^2q}{6\,mn^2q}$.

153. To find the quotient of two monomials:

1. *Find the sign of the quotient,* by the rule of signs, Sec. 151.

2. *Find the numerical coefficient,* by dividing the given numerical coefficients.

3. *Find the literal part,* by dividing the given literal parts.

Thus, in $18\,a^3b \div -3\,ab$, the sign is $-$, the numerical coefficient is $18 \div 3$ or 6, and the literal part is $a^3b \div ab$ or a^2. $\therefore$ the quotient is $-6\,a^2$.

WRITTEN EXERCISES

Divide:

1. $\dfrac{2\,a^2cx^2y}{-2\,axy}$.
2. $\dfrac{-\,abcx^2y}{-\,cx}$.
3. $\dfrac{-5\,abc^2}{-5\,bc}$.
4. $\dfrac{10\,a^3x^2y^3}{-5\,ax^2}$.
5. $\dfrac{10\,m^3n^3}{-\frac{1}{6}\,mn^2}$.
6. $\dfrac{1.4\,a^5b}{.7\,a^3b}$.
7. $\dfrac{a^{7m}b^{4c}}{a^{3m}b^c}$.
8. $\dfrac{18\,a^3b^3}{3\,a^2b}$.
9. $\dfrac{24\,abc}{-3\,b}$.
10. $\dfrac{1.5\,x^3yz}{-5\,x^2y}$.
11. $\dfrac{m^2n^2p^2}{-n\cdot -p}$.
12. $\dfrac{-a^4b^3c^3}{-a^2\cdot -bc}$.
13. $\dfrac{-35\,b^6c^6}{7\,b^3c^3}$.
14. $\dfrac{10^{6n}x^{3n+8}}{10^{2n}x^{3n+4}}$.
15. $\dfrac{-2\,a^3x^3}{\frac{1}{2}\,ax^2}$.
16. $\dfrac{-2\,p^3q^4r^4t^4}{-\frac{1}{5}\,pqrt^3}$.
17. $\dfrac{a^4b^2c^2x}{-a^2bcx}$.
18. $\dfrac{-16\,a^2x^2y^3}{4\,y}$.
19. $\dfrac{27\,xy^2z^4}{-3\,z^3}$.
20. $\dfrac{16\,b^6cx^6}{-4\,b^3cx^3}$.
21. $\dfrac{54\,m^2p^3x^5}{-9\,mp^2x^3}$.
22. $\dfrac{a^{n+7}b^{3n+6}c^{2n+9}}{-a^4b^3c^7}$.
23. $3\,a^4b^4c^3 \div -\,abc$.
24. $2.5\,a^2x^3y \div -5\,ax^2y$.
25. $55\,a^2x^2 \div -11\,a^2x$.
26. $\dfrac{-18\,x^{a+5}y^{2a+7}}{3\,x^4y^{a+3}}$.
27. $\frac{4}{3}\,\pi r^3 \div \frac{1}{3}\pi r$.
28. $\frac{1}{6}\,\pi d^3 \div \frac{1}{2}\,\pi d^2$.
29. $-27\,x^2y^2z^2 \div 3\,xyz^2$.

DIVISION OF POLYNOMIALS

154. Preparatory.

1. $2\overline{)\;4 \text{ bu. } 2 \text{ pk.}}$ — ( ) bu. ( ) pk. $2\overline{)\;4 \text{ lb. } 2 \text{ oz.}}$ — ( ) lb. ( ) oz. $2\overline{)\;4\,l + 2\,z}$ — $(\;)l + (\;)z$

2. $3\overline{)\;9 \text{ ft. } 6 \text{ in.}}$ — ( ) ft. ( ) in. $3\overline{)\;9 \text{ mi. } 6 \text{ rd.}}$ — ( ) mi. ( ) rd. $3\overline{)\;9\,m + 6\,r}$ — $(\;)\,m + (\;)\,r$

Multiplication	Division
Since (1)	then, (2)
$\begin{array}{r} a - bd + 5\,c \\ m \\ \hline ma - mbd + 5\,mc \end{array}$	$\begin{array}{r} m\overline{)\,ma - mbd + 5\,mc} \\ a - bd + 5\,c \end{array}$

155. Accordingly,

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial, and use the signs obtained as the signs of the quotient.

WRITTEN EXERCISES

Divide and test:

1. $(6a^2 + 3a) \div 3$.
2. $(12ab + 4b) \div 4$.
3. $(6a^2 + 3a) \div a$.
4. $(12ab + 5b) \div b$.
5. $(6a^2 + 3a) \div 3a$.
6. $(12ab + 4b) \div 4b$.
7. $5a^2 - 4ab + 4a$ by a.
8. $a^6 - 5a^5 + 3a^4$ by a^2.
9. $(x^2y + xy^2) \div xy$.
10. $(3x^2y - 6xy) \div 3xy$.
11. $25a^2 + 10ab + 5b^2$ by 5.
12. $27a^2b - 9ab^2 + 9a^2b^2$ by $9ab$.
13. $(12mn + 27mn^2p) \div 3mn$.
14. $(6x^2y - 4x^2z + 6xyz) \div 2x$.
15. $4x^4y^4 - 8x^3y^3 + 6xy^3$ by $-2xy$.
16. $-3a^2 + \frac{9}{2}ab - 3ac$ by $-\frac{3}{2}a$.
17. $x^3y - 3x^2y + 9xy^2$ by $3xy$.
18. $(x^3y - 3x^2y^3 + 9xy^2) \div 3xy$.
19. $(a^2c^2 - 2abc^2 + 3ac^3) \div ac^2$.
20. $a^2c^2 - 2abc^2 + 3ac^3$ by $-ac^2$.
21. $(5a^3b^3 - 35a^2b^2c^2 + 2ab^2c^4) \div 5ab$.
22. $(2m^2n^2 - 3mn^3 + 4m^2n - n^4) \div 3n$.
23. $(a^3x^3y - 3a^2bx^2y + 3ab^2xy^2 - a^2b^3xy^3) \div axy$.
24. $x^{7n} + 4x^{9n}y^2 + 3x^{8n}y^{2n}$ by x^{3n}.
25. $a^{7c+5} + a^{4c+8} + a^{11c+12}$ by a^4.
26. $t^{3x+6} - 6t^{6x+3} + 3t^{3x+3}$ by t^{x+1}.
27. $10^{4n+12} + 10^{8n+9} + 10^{16n+6}$ by 10^{4n+3}.

156. To Divide by a Polynomial. This process is seen best from examples.

1. Divide $x^2+3xy+2y^2$ by $y+x$.

Arrange the terms of the divisor and the dividend according to the powers of the same letter (x in this example).

Divide the first term of the dividend by the first term of the divisor.

The result (in this example, x) is the first term of the quotient.

Multiply the entire divisor by this term and subtract.

Divide the first term of the remainder by the first term of the divisor. The result (in this example, $2y$) is the second term of the quotient.

Multiply the entire divisor by this term and subtract.

Continue the process until a remainder zero is reached.

Problems in which such a remainder can not be reached will be treated later.

$$\begin{array}{r l}
 & \text{QUOTIENT} \\
 & x+2y \\
\text{DIVISOR } x+y) & x^2+3xy+2y^2 \\
 & \underline{x^2+\ \ xy\qquad} \\
 & 2xy+2y^2 \\
 & \underline{2xy+2y^2}
\end{array}$$

Test. Multiply the quotient by the divisor. If the work has been correctly done (including the multiplication), the result will equal the dividend. Or, substitute arbitrary values and proceed as in previous cases.

2. Divide $6a^3-17a^2b+16b^3$ by $3a-4b$.

SOLUTION

$$\begin{array}{r l}
 & 2a^2-3ab-4b^2 \\
3a-4b) & 6a^3-17a^2b+16b^3 \\
 & \underline{6a^3-8a^2b} \\
 & -9a^2b \\
 & \underline{-9a^2b+12ab^2} \\
 & -12ab^2+16b^3 \\
 & \underline{-12ab^2+16b^3}
\end{array}$$

TEST (Substitute $a=b=1$)

Divisor × Quotient = Dividend

$$(3-4)(2-3-4)=6-17+16$$
$$-1\cdot(-5)=5$$
$$5=5$$

WRITTEN EXERCISES

Divide and test:

1. $a^2+2ab+b^2$ by $a+b$.
2. a^2+3a+2 by $a+1$.
3. $x^2-11x+30$ by $x-5$.
4. $x^2+4xy+4y^2$ by $x+2y$.
5. $3c^2+7cd+2d^2$ by $d+3c$.
6. $6a^2-7a-3$ by $2a-3$.

7. $2x^2 - xy - 3y^2$ by $x + y$.

8. $3a^2 + ab - 2b^2$ by $3a - 2b$.

9. $6m^2 + mn - 2n^2$ by $3m + 2n$.

10. $12y^2 + 19y - 21$ by $3y + 7$.

11. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.

12. $96a^2 - 4ab - 15b^2$ by $12a - 5b$.

13. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$.

14. $a^4 + a^2b^2 + b^4$ by $a^2 - ab + b^2$.

15. $x^3 - 2x^2 - 2x + 1$ by $x + 1$.

16. $3p^2 - 115 - 8p$ by $p + 5$.

17. $3a^4 - 2a - 1865$ by $a - 5$.

18. $14y^4 - 13y - 43y^2 + 32y^3 + 3$ by $7y^2 - 5y - 3$.

19. $21a^3 - 4a^2 - 16 - 46a$ by $4a + 2 - 3a^2$.

20. $\dfrac{1 + a^6}{1 - a^2 + a^4}$.

21. $\dfrac{a^8 - b^8}{a^2 + b^2}$.

22. $\dfrac{a^3 + b^3}{a^2 - ab + b^2}$.

23. $\dfrac{a^3 - b^3}{a^2 + ab + b^2}$.

24. $\dfrac{x^6 - 4x^4y^2 + 4x^2y^4 - y^6}{x^2 - y^2}$.

25. $\dfrac{x^{2p} + 8x^p + 16}{x^p + 4}$.

26. $\dfrac{9m^2 + 12mn + 4n^2}{3m + 2n}$.

27. $\dfrac{a^4 + 4b^4}{a^2 + 2ab + 2b^2}$.

28. $\dfrac{6a^2b^2 - ab^3 - 12b^4}{3ab + 4b^2}$.

29. $\dfrac{4x^4y^4 + 1}{2x^2y^2 - 2xy + 1}$.

30. $\dfrac{a^5 - a^4 - 11a^3 + 16a^2 - 2a - 3}{a^2 - 4a + 3}$.

31. $\dfrac{6x^4 + 17x^3 - 19x - 4}{2x^3 + 3x^2 - 4x - 1}$.

32. $\dfrac{x^{2n} + 3x^ny^{2n} - x^ny^n - 3y^{3n}}{x^n - y^n}$.

157. In algebra, as in arithmetic, the division may not be exact. That is, no remainder may be zero, however far the division is carried.

Thus, in the example at the right, there is a remainder, 2. The division might be continued, the next term of the quotient being $\frac{2}{x}$; but it is customary to stop as soon as a remainder is reached that is of lower degree than the divisor. The integral part of the quotient has now been found; it is called the integral quotient.

$$\begin{array}{r|l} & x^3 - x^2 + x - 1 \\ \hline x+1 & x^4 + 1 \\ & \underline{x^4 + x^3} \\ & \quad - x^3 \\ & \quad \underline{- x^3 - x^2} \\ & \qquad x^2 \\ & \qquad \underline{x^2 + x} \\ & \qquad\quad - x + 1 \\ & \qquad\quad \underline{- x - 1} \\ & \qquad\qquad 2 \end{array}$$

By using the fractional form to indicate the division of the remainder, the result of the above division may be expressed thus:

$$\frac{x^4+1}{x+1} = x^3 - x^2 + x - 1 + \frac{2}{x+1}.$$

The right member of the equation is called the complete quotient.

Test. Dividend = divisor × integral quotient + the remainder.

Substituting $x = 1$.

Dividend = Divisor × Integral Quotient + Remainder.

$$1 + 1 = (1 + 1)(1 - 1 + 1 - 1) + 2.$$
$$2 = 2 \cdot 0 + 2.$$

WRITTEN EXERCISES

Find the quotient and remainder:

1. $(x^3 - 1) \div (x + 1)$.
2. $(a^2 + x^2) \div (a + x)$.
3. $\dfrac{x^5 + x + 1}{x^2 - 1}$.
4. $\dfrac{x^4 - 2x^2 + 1}{x^2 + 1}$.
5. $(3x^2 - 5x + 2) \div (x - 4)$.
6. $(8y^2 + 7y - 1) \div (2y + 3)$.
7. $\dfrac{6ax - 9ay - 4bx + 8by}{3a - 2b}$.
8. $\dfrac{2a^3 - 2a^2 - 6a + 4}{2a - 3}$.

SUMMARY

I. Definitions and Laws.

1. *Division* is the process of finding one of two factors when their product and the other factor are given. Sec. 147.

2. *Law of Exponents in division:*

$$a^{m+r} \div a^r = a^m. \qquad \text{Sec. 148.}$$

3. The quotient is not altered, if both dividend and divisor are multiplied or divided by the same number. Sec. 151.

4. *Law of signs in division:* When the signs of the dividend and the divisor are alike, the quotient has the sign plus; and when the signs are unlike, the quotient has the sign minus. Sec. 152.

II. **Processes.**

1. *To find the quotient of two monomials*, divide the coefficient of the dividend by the coefficient of the divisor, and to this quotient annex each letter with an exponent equal to the exponent of that letter in the dividend, diminished by its exponent in the divisor, omitting any letter having the same exponent in the dividend and divisor. Secs. 149, 153.

2. *To divide a polynomial by a monomial*, divide each term of the polynomial by the monomial and use the signs obtained as the signs of the quotient. Sec. 155.

3. *To divide by a polynomial*, arrange the expressions according to the powers of the same letter and divide step by step as explained in Sec. 156.

4. The product of the divisor and the integral quotient plus the remainder must equal the dividend. Sec. 150.

REVIEW

ORAL EXERCISES

1. Find the numbers to fill the blanks:

	(1)	(2)	(3)	(4)	(5)
Dividend:	——	——	$-15\,x^2y^3$	$-21\,mp$	$-27\,a^3b$
Divisor:	$3\,a$	$9\,xy$	——	$7\,m$	$-\ 9\,a^2b$
Quotient:	$4\,b^2$	$3\,xz$	$5\,xy^2$	——	——

State the quotients:

2. $4\,a \div 6\,a.$

3. $-ab \div b.$

4. $\dfrac{-12cx}{-3c}.$

5. $\dfrac{14\,x^3}{7\,x}.$

6. $\dfrac{-26\,ab^4}{-13\,b^2}.$

7. $\dfrac{30\,m^2t^2}{-5\,m^2t^2}.$

8. $\dfrac{-24\,abc^3}{-8\,bc}.$

9. $\dfrac{24\,y^5}{-3\,y^2}.$

10. $\dfrac{(a+x)^2}{a+x}.$

11. $\dfrac{9\,m^2-25}{3\,m-5}.$

12. $\dfrac{-32\,ax^4}{4\,x^3}$.

13. $\dfrac{t^2-14\,t+49}{t-7}$.

14. $\dfrac{1-16\,y+64\,y^2}{1-8\,y}$.

15. $\dfrac{ax^2+bx^2}{x^2}$.

16. $\dfrac{x^2-y^2}{x-y}$.

17. $\dfrac{x^2-y^2}{x+y}$.

18. $\dfrac{a^2-x^2}{a+x}$.

19. $\dfrac{m^2p^2-n^2q^2}{mp+nq}$.

20. $\dfrac{x^4-y^4}{x^2-y^2}$.

21. $\dfrac{m^4-n^4}{m^2+n^2}$.

22. $\dfrac{x^2y^2-m^2n^2}{xy-mn}$.

23. $\dfrac{x^{2m}-y^{2r}}{x^m-y^r}$.

WRITTEN EXERCISES

Divide and test:

1. $\dfrac{16\,a^3x}{8\,ax}$.

2. $\dfrac{-42\,m^6y^3}{6\,m^2y}$.

3. $\dfrac{144\,a^5x^4}{-16\,ax^3}$.

4. $\dfrac{-14641\,p^2q}{-11\,p}$.

5. $(14\,ax^2+6\,ax)\div 2\,a$.

6. $(25\,t^3-40\,t)\div -5$.

7. $(18\,x^2z^3-24\,x^3z^2)\div -6\,xz^2$.

8. $(a^2bc-ab^2c+2\,abc^2)\div abc$.

9. $(-27\,m^{2x}y^{2a}-9\,m^{5x}y^{3a})\div 9\,m^xy^a$.

10. $(-25\,p^{2a}q^{3r}+10\,p^{3a}q^{2r})\div 5\,p^aq^r$.

Find the quotient and remainder, if any:

11. $\dfrac{x^5-1}{x^2-1}$.

12. $\dfrac{x^3-7\,x^2+3\,x-1}{x-3}$.

13. $\dfrac{x^2+xzy^2-zy+z^2}{x+y+z}$.

14. $\dfrac{a^4-5\,a^2+4}{a+2}$.

15. $\dfrac{x^3-2\,x^2+x-4}{x-3}$.

16. $\dfrac{x^4-2\,x^2+3\,x}{x^2-1}$.

17. $\dfrac{8-12\,a+6\,a^2-a^3}{2-a}$.

18. $\dfrac{15\,a^2b+6\,ab^2+8\,a^3+3\,b^3}{8\,a-b}$.

19. $\dfrac{-7\,r^2+3\,r^3+5+r^4}{r^2-5\,r+2}$.

20. $\dfrac{18\,x^4-24\,x^3+38\,x^2-68\,x+32}{3\,x-2}$.

SUPPLEMENTARY WORK

Division by Detached Coefficients

When both divisor and dividend involve a regular series of powers of the same letters, it is easier to divide with *coefficients only.*

Thus to divide $x^3 - 4x^2 + 4x - 1$ by $x - 1$.

WORK IN FULL

$$
\begin{array}{r|l}
 & x^2 - 3x + 1 \\
\hline
x - 1 & x^3 - 4x^2 + 4x - 1 \\
 & \underline{x^3 - x^2} \\
 & -3x^2 \\
 & \underline{-3x^2 + 3x} \\
 & x - 1 \\
 & \underline{x - 1}
\end{array}
$$

WORK WITH COEFFICIENTS ONLY

$$
\begin{array}{r|l}
 & 1 - 3 + 1 \text{ or } x^2 - 3x + 1 \\
\hline
1 - 1 & 1 - 4 + 4 - 1 \\
 & \underline{1 - 1} \\
 & -3 \\
 & \underline{-3 + 3} \\
 & 1 - 1 \\
 & \underline{1 - 1}
\end{array}
$$

TEST. Letting $x = 2$,

$$(2-1)(4-6+1) = 8-16+8-1.$$
$$-1 = -1.$$

If the series of powers is not complete, zero coefficients must be used.

Thus, $x^3 + 3x^2 + 1$ must be regarded as $x^3 + 3x^2 + 0x + 1$, and the coefficients $1 + 3 + 0 + 1$ must be used in the division.

WRITTEN EXERCISES

Divide by detached coefficients and test:

1. $(x^2 + 5x + 6) \div (x + 2)$.
2. $(x^2 - 2x - 3) \div (x - 3)$.
3. $10 + x^2 - 7x \div (x - 2)$.
4. $(x^4 - 1) \div (x^2 - 1)$.
5. $(x^3 + 3x^2 + 3x + 1) \div (x + 1)$.
6. $(y^3 - 3y^2 + 3y - 1) \div (y - 1)$.
7. $(6x + 1 + 9x^2) \div (3x + 1)$.
8. $(a^4 + a^2 + 1) \div (a^2 - a + 1)$.
9. $\dfrac{a^4 - 1}{a - 1}$.
10. $\dfrac{a^3 - 1}{a^2 + a + 1}$.
11. $\dfrac{8x^3 + 1}{2x + 1}$.
12. $\dfrac{m^4 + 4m^3 + 6m^2 + 4m + 1}{m^2 + 2m + 1}$.

Special Quotients

The general form of quotients represented by the type $(x^n \pm y^n) \div (x \pm y)$ may readily be seen by division.

Thus, considering $(x^n + y^n) \div (x + y)$, we have

$$
\begin{array}{r|l}
x + y & \underline{x^{n-1} - x^{n-2}y + x^{n-3}y^2 \dots} \\
 & x^n \qquad\qquad + y^n \\
 & \underline{x^n + x^{n-1}y} \\
 & \quad - x^{n-1}y \\
 & \quad \underline{- x^{n-1}y - x^{n-2}y^2} \\
 & \qquad\qquad + x^{n-2}y^2
\end{array}
$$

By inspection, what do you think the fourth term of the quotient will be? The fifth? Verify your opinion by continuing the above division.

WRITTEN EXERCISES

Find the quotient and remainder if any:

1. $(x^5-1)\div(x-1)$.
2. $(x^6+1)\div(x-1)$.
3. $(x^5+1)\div(x+1)$.
4. $(x^6-1)\div(x+1)$.
5. $(x^4-y^4)\div(x+y)$.
6. $(x^4-y^4)\div(x-y)$.
7. $(x^5+y^5)\div(x+y)$.
8. $(x^7+y^7)\div(x+y)$.
9. $\dfrac{x^7-y^7}{x-y}$.
10. $\dfrac{x^7+y^7}{x-y}$.
11. $\dfrac{x^6-y^6}{x-y}$.
12. $\dfrac{x^6+y^6}{x-y}$.
13. $\dfrac{x^6-y^6}{x+y}$.
14. $\dfrac{x^6+y^6}{x+y}$.

15. Substitute 5 for n in the above division of the general type, and compare with the result of Exercise 7.

16. Substitute 6 for n similarly, and compare with the result of Exercise 14.

17. Substitute 7 for n, and compare with Exercise 8.

18. Take $n = 5$ and $y = 1$, and compare with Exercise 3.

Discuss similarly the following type quotients:

19. $(x^n + y^n) \div (x - y)$.

20. $(x^n - y^n) \div (x + y)$.

21. $(x^n - y^n) \div (x - y)$.

Table of Results. The following table shows for what values of n the division is exact in the type $(x^n \pm y^n) \div (x \pm y)$:

$a.$ $(x^n + y^n) \div (x + y)$, if n is odd.
$b.$ $(x^n + y^n) \div (x - y)$, for no n.
$c.$ $(x^n - y^n) \div (x + y)$, if n is even.
$d.$ $(x^n - y^n) \div (x - y)$, for every n.

ORAL EXERCISES

Name a factor of each of the following:

1. $x^5 - y^5$.
2. $x^7 + a^7$.
3. $x^8 - b^8$.
4. $1 - x^6$.
5. $x^9 + a^9$.
6. $x^{10} - 1$.
7. $x^{13} - y^{13}$.
8. $x^{13} + y^{13}$.
9. $a^{14} - b^{14}$.
10. $32 x^5 - 1$, or $(2x)^5 - 1$.
11. $16 x^4 - 81$.

The factor of the first degree having been found by inspection, the other is a regular series of powers that can be written directly; it has alternate signs in the cases (a) and (c), and plus signs in case (d).

For example:

$$x^7 + a^7 = (x + a)(x^6 - ax^5 + a^2x^4 - a^3x^3 + a^4x^2 - a^5x + a^6).$$

$$\begin{aligned} 32a^5 - 1 &= (2a)^5 - 1 \\ &= (2a - 1)[(2a)^4 + (2a)^3 \cdot 1 + (2a)^2 \cdot 1^2 + 2a \cdot 1^3 + 1^4] \\ &= (2a - 1)(16a^4 + 8a^3 + 4a^2 + 2a + 1). \end{aligned}$$

TEST. $32 - 1 = 1(16 + 8 + 4 + 2 + 1)$.

$x^{10} - r^{10}$ has both $x - r$ and $x + r$ as factors, but it is better first to factor as a difference of two squares.

$$\begin{aligned} x^{10} - r^{10} &= (x^5)^2 - (r^5)^2 = (x^5 - r^5)(x^5 + r^5) \\ &= (x - r)(x^4 + rx^3 + r^2x^2 + r^3x + r^4) \cdot (x + r)(x^4 - rx^3 + r^2x^2 \\ &\qquad - r^3x + r^4). \end{aligned}$$

WRITTEN EXERCISES

Factor:

1. $32a^5 - b^5$.
2. $128a^7 + 1$.
3. $32 - 243t^5$.
4. $(c - d)^7 + 1$.
5. $a^7 - b^7$.
6. $1 - m^{10}$.
7. $x^6 - a^6$.
8. $1 - (a + b)^4$.
9. $(x - 3y)^5 + 1$.
10. $x^8 - y^8$ (four factors).
11. $(a + b)^5 - (a - b)^5$.
12. $(x - y)^4 - (x + y)^4$.

CHAPTER X

EQUATIONS

ONE UNKNOWN

158. An equation involving one unknown to the first degree is solved by the properties of the equation. Sec. 20, p. 11.

EXAMPLES

1. Solve: $2x - 7 = -15.$ (1)

Adding 7 to both members of (1), $2x = -8.$ (2)

Dividing both members of (2) by 2, $x = -4.$ (3)

TEST. $2 \cdot -4 - 7 = -15.$

2. Solve: $4t + 7 = 6t - 31.$ (1)

Subtracting 7 from both members of (1), $4t = 6t - 38.$ (2)

Subtracting $6t$ from (2), $-2t = -38.$ (3)

Dividing (3) by -2, $t = 19.$ (4)

TEST.

$$4 \cdot 19 + 7 = 6 \cdot 19 - 31.$$
$$76 + 7 = 114 - 31.$$
$$83 = 83.$$

ORAL EXERCISES

Solve:

1. $4x = 8.$
2. $5x = -15.$
3. $-6x = 18.$
4. $-7x = -28.$
5. $y + 3 = 5.$
6. $y - 2 = 7.$
7. $y - 8 = -10.$
8. $y + 8 = -5.$
9. $2t - 8 = 0.$
10. $4t + 12 = 0.$
11. $6 - 3t = 0.$
12. $18 + 6t = 0.$

WRITTEN EXERCISES

Solve:

1. $3x + 5 = 26.$
2. $4x + 7 = -5.$
3. $2 - x = 3 + x.$
4. $5t + 7 = -8.$
5. $6t + 9 = 2t - 27.$
6. $3 - 2t = 3t - 5.$

7. $\frac{x}{3}+5=9.$

8. $\frac{x}{5}+3=0.$

9. $\frac{2x}{7}+1=-4.$

10. $15p+29=4p-4.$

11. $1-7w=8w+16.$

12. $4s-7=3-s.$

13. $2x+11-7x=4x.$

14. $5w-13+8w=17.$

15. $4x+8=2-9x-10.$

16. $x+a=2x.$

17. $3t-a=5a.$

18. $ay+5a=8a.$

19. $5(x-4)=6(x+1).$

20. $-3(x+7)=2(1-3x).$

21. $a(x-b)=3ab.$

22. $(x-4)(x+4)=x^2-8x.$

23. $(x-1)^2=(x-3)^2.$

24. $(x-5)(x+8)=(x-7)^2.$

25. $y(9y-5)=(3y+1)^2.$

26. $t^2-1=(t+4)^2.$

159. Equations may be used in solving business problems.

EXAMPLE

1. A commission merchant remitted \$475 as the proceeds of a sale of 500 bu. of potatoes after deducting his commission of 5%. For how much did he sell the potatoes?

SOLUTION. 1. Let x = the number of dollars received for the potatoes.
2. Then, $x-.05x$ = the amount remitted.
3. $\therefore x-.05x=475$, according to the problem.
4. $\therefore .95x=475.$
5. Dividing by .95, $x=500.$
6. $\therefore$ the potatoes sold for \$500.

TEST. $500-.05\cdot 500=475.$

This merely tests the correctness of the work after step 3; it does not test the correctness of the equation in step 3; to do this the result, 500, must be tested *in the conditions of the problem itself.*

Thus, (1) 5% of \$500 = \$25, (2) \$500 − \$25 = \$475, the proceeds. If equation 3 had been incorrectly written, the result found in step 6 might have been a correct solution of equation 3 without giving the correct proceeds.

WRITTEN EXERCISES

Solve and test:

1. An auctioneer was paid 2 % of the amount sold; the proprietor realized $1470. For how much were the goods sold?

2. A dealer sold a refrigerator for $22 at a gain of 10 %. What did it cost the dealer?

3. A merchant sold a damaged fur coat for $51 at a loss of 15 %. What did the coat cost him?

4. A lawyer remitted $760 after deducting a fee of 5 % for collecting a debt. How much did he collect?

5. The amount of a certain principal at 4 % simple interest for 2 years is $220. What is the principal?

6. The total amount of insurance in force in New York City and Buffalo in a recent year was $2,700,000,000; Buffalo had $\frac{2}{25}$ as much as New York. How much had each?

7. The trade route from San Francisco to New York will be 5240 mi. via the Panama Canal; this will be 62 % shorter than the route by Cape Horn. What is the length of the latter route?

8. The trade route from Yokohama to New Orleans will be 9250 mi. by the Panama route; this will be 38 % shorter than the route by Cape Horn. What is the length of the latter?

9. Japan recently gave American manufacturers an order for 2000 cars and locomotives; they ordered 19 times as many cars as locomotives. How many of each were ordered?

10. The amount of condensed milk produced by New York and Illinois is $\frac{3}{4}$ of that produced by the rest of the country; the total amount produced in the country annually is about 154 million pounds. How many pounds are produced by the two states together?

11. The United States produces 3 times as much cotton as the rest of the world; the total cotton production in a recent year was 14 million bales. What was the number of bales produced by the United States?

12. North Carolina, South Carolina, and Georgia produced together 3 million bales; South Carolina produced twice as much as North Carolina and $\frac{1}{3}$ as much as Georgia. How many bales did each produce?

13. Mississippi and Texas together produced 4 million bales; Texas produced $1\frac{2}{3}$ times as much as Mississippi. How many bales did each produce?

14. The United States consumes $\frac{2}{5}$ as much cotton as does the rest of the world. How many bales is this, when the whole world consumes 14 million bales annually?

15. Three electric lights have together a strength of 280 candle power. The second has 40 candle power more than the first, and the third double that of the first. Find the strength of each.

TWO UNKNOWNS

160. Problems involving more than one unknown often require more than one equation in their solution.

EXAMPLES

1. A street car of a certain make has 48 seats, some single and some double; the seating capacity of the car is 56. How many single seats are there? How many double seats?

SOLUTION. Let $x =$ the number of single seats

and $y =$ the number of double ones.

Then, $\begin{cases} x + y = 48, & (1) \\ x + 2y = 56. & (2) \end{cases}$

Subtracting equation (1) from equation (2), $y = 8$. (3)

Substituting this value of y in equation (1), $x + 8 = 48$. (4)

$\therefore x = 40$. (5)

Therefore there were 40 single seats and 8 double ones. (6)

TEST.

$$40 + 8 = 48.$$
$$40 + 2 \cdot 8 = 56.$$

2. A salesman sold 15 suits for \$164; some of the suits were sold for \$10 and the others for \$12. How many were there of each kind?

SOLUTION. Let x = the number of suits at \$10
and y = the number of suits at \$12.

Then, $\begin{cases} x + y = 15, & (1) \\ 10x + 12y = 164. & (2) \end{cases}$

Multiplying (1) by 10, $10x + 10y = 150$. (3)

Subtracting (3) from (2), $2y = 14$. (4)

$\therefore y = 7$. (5)

Substituting 7 for y in (1), $x = 8$. (6)

Therefore, there were 8 suits at \$10, and 7 suits at \$12.

TEST. $8 + 7 = 15$.
$10 \cdot 8 + 12 \cdot 7 = 164$.

In step (3), equation (1) was multiplied by 10 to make the coefficient of x the same as that in equation (2). Then, by subtracting equation (3) from (2), an equation in one unknown resulted. All sets of two equations with two unknowns can be solved in a similar way. See steps (3) and (4) of the following example.

3. Salesmen A and B sell hats at two prices. A sells 10 of the first kind and 5 of the second, and his sales amount to \$45; B sells 6 of the first kind and 8 of the second, and his sales amount to \$47. Find the price of each grade of hat.

SOLUTION. Let x = price of first kind of hat
and y = price of second kind.

Then, $\begin{cases} 10x + 5y = 45, & (1) \\ 6x + 8y = 47. & (2) \end{cases}$

Multiplying (1) by 3, $30x + 15y = 135$. (3)

Multiplying (2) by 5, $30x + 40y = 235$. (4)

Subtracting (3) from (4), $25y = 100$. (5)

$y = 4$. (6)

Substituting in (1) or (2), $x = 2\frac{1}{2}$.

Supply the test.

WRITTEN EXERCISES

Solve for x and y:

1. $x + y = 4$,
$3x + 5y = 18$.

2. $x + y = 7$,
$3x + 10y = 42$.

3. $2x+y=9,$
$x+2y=12.$

4. $7x+y=14,$
$x+9y=64.$

5. $2x+3y=22,$
$5x+4y=48.$

6. $2x+6y=34,$
$6x+8y=62.$

7. $2x+4y=38,$
$3x+y=27.$

8. $x+10y=11,$
$\frac{1}{2}x+\frac{1}{5}y=\frac{7}{10}.$

9. A salesman sells suits at $12 and $15. He sells 12 suits for $165. How many does he sell at each price?

10. Solve the same problem for the following numbers:

	(1)	(2)	(3)	(4)	(5)
Price of first kind of suit	$14	$25	$15	$16	$28
Price of second kind of suit	$18	$40	$25	$20	$35
Total number of suits sold	8	18	16	12	13
Total amount of sales	$132	$570	$300	$220	$420

11. If 15 bicycles and 10 tricycles are worth $320, and 10 bicycles and 6 tricycles are worth $318, find the price of each.

12. Solve the same problem for the following numbers:

	(1)	(2)	(3)
Number of bicycles	8, 5	6, 3	5, 1
Number of tricycles	3, 7	2, 7	2, 8
Value	$188, $220	$180, $270	$154, $274

161. In the work of solving the preceding problems the process has always been to combine the given equations so as to obtain an equation involving only one of the unknowns.

The other unknown is said to have been **eliminated.**

162. Independent Equations. Equations which express different relations between the same unknowns are called **independent equations.**

Thus, $x - y = 1$ and $x + y = 7$ are independent equations; for one expresses the *difference* between two numbers, x and y, while the other expresses the *sum* of the same two numbers.

$x - y = 1$ and $2x - 2y = 2$ are not independent, for the second when divided by 2 is the same as the first.

163. Simultaneous Equations. Two or more equations are said to be **simultaneous** when all of them are satisfied by the same values of the unknowns.

164. Systems of Equations. Two or more equations considered together are called a **system** of equations.

Thus, each of the exercises of p. 116 contains a system of equations.

165. Equations in which the unknowns are involved to the first degree only are called **equations of the first degree.**

166. Method of Addition and Subtraction. All systems of two independent simultaneous equations of the first degree in two unknown quantities can be solved by the method used in the problems above, which has been called the **method of addition and subtraction.**

167. *The method of addition and subtraction consists in multiplying one or both of the given equations by such numbers that the coefficients of one of the unknowns become equal. Then by subtraction this unknown is eliminated, and the solution is reduced to that of a single equation.*

If the coefficients of one unknown are made numerically equal, but have opposite signs, the equations should be added.

EXAMPLE

Solve: $\begin{cases} 4x - 3y = 5, & (1) \\ 6x + 2y = 14. & (2) \end{cases}$

Multiplying (1) by 2,	$8x - 6y = 10,$	(3)
and (2) by 3,	$18x + 6y = 42.$	(4)
Adding (3) and (4),	$26x = 52,$	(5)
and	$x = 2.$	(6)
Substituting 2 for x in (2),	$y = 1.$	(7)

Test as usual.

WRITTEN EXERCISES

Solve and test:

1. $2x + y = 5,$
   $x - y = 1.$
2. $3x + 2y = 7,$
   $x - 2y = 3.$
3. $2x + y = 1\frac{1}{2},$
   $x - y = 0.$
4. $3x + 2y = 2,$
   $x + y = \frac{5}{6}.$
5. $x + 5y = 35,$
   $5x + y = 31.$
6. $4x + 3y = 18,$
   $2x + 2y = 10.$
7. $x + 2y = 0,$
   $4x - 3y = 4.$
8. $2x + 3y = 4,$
   $3x + 2y = 1.$
9. $\frac{1}{2}x + \frac{1}{3}y = 4,$
   $\frac{3}{4}x + \frac{2}{3}y = 5.$
10. $.3x + .2y = .1,$
    $.2x + .3y = .4.$
11. $4x + y = 34,$
    $4y + x = 16.$
12. $4x - y = 7,$
    $3x + 4y = 29.$
13. $2x + 3y = 4,$
    $3x - 2y = -7.$
14. $2x + 3y - 8 = 0,$
    $7x - y - 5 = 0.$

15. In a recent year the value of the hay crop in the United States exceeded that of the cotton crop by \$30,000,000; the two amounted to \$1,180,000,000. Find the value of each.

16. The value of the cotton crop exceeded that of the wheat crop by \$ 50,000,000; the two amounted to \$ 1,100,000,000. What was the value of each?

17. If the value of the eggs marketed had been increased by $\frac{1}{104}$ of itself, it would have equaled the value of the wheat crop; the two amounted to \$ 1,045,000,000. What was the value of the eggs marketed?

18. If the value of the sugar produced had been increased $\frac{4}{25}$ of itself, it would have equaled the value of the barley crop; the two amounted to \$ 108,000,000. What was the value of each?

19. The value of the oat crop exceeded that of the potato crop by \$ 144,000,000; the two amounted to \$ 420,000,000. Find the value of each.

168. Method of Substitution. The method of addition and subtraction can be used to solve all systems of two simultane-

ous equations with two unknowns, but occasionally problems occur in which other methods are shorter. The most useful of these is the **method of substitution.**

EXAMPLES

1. Solve: $\begin{cases} 3x+2y=6, & (1)\\ 5x-\quad y=1. & (2)\end{cases}$

From equation (2), $y = 5x - 1.$ (3)

Substituting in (1), $3x + 10x - 2 = 6.$ (4)

$\therefore 13x = 8,$ (5)

and $x = \frac{8}{13}.$ (6)

From (6) and (3), $y = \frac{27}{13}.$ Verify. (7)

2. Solve: $\begin{cases} 4x-3y=9, & (1)\\ 5x=11. & (2)\end{cases}$

From equation (2), $x = \frac{11}{5}.$ (3)

Substituting in (1), $\frac{44}{5} - 3y = 9.$ (4)

$\therefore y = -\frac{1}{15}.$ Verify. (5)

169. *The method of substitution consists in expressing one unknown in terms of the other by means of one equation and substituting this value in the other equation, thus eliminating one of the unknowns.*

This may be the shorter method when an unknown in either equation has the coefficient $0, +1$, or -1.

WRITTEN EXERCISES

Solve by substitution:

1. $x+y=75,$ $3x-3y=15.$
2. $5x+2y=31,$ $x=12y.$
3. $3x+2y=12,$ $x+y=5.$
4. $x+2y=7,$ $x=\frac{3}{2}y.$
5. $x+3y=11,$ $3x+y=9.$
6. $x-4y=8,$ $x+2y=14.$
7. $y+15x=53,$ $x+3y=27.$
8. $5x-7y=-35\frac{1}{2},$ $2x-y=\frac{5}{7}.$
9. $3x-5y=31,$ $4y=-10x.$
10. $x+y=\frac{7}{6},$ $x+7y=\frac{31}{6}.$
11. $x+y=12,$ $3x+y=24.$
12. $x+2y=10,$ $5x-2y=2.$

13. $2x - y = 1,$
$3y - x = 2.$

14. $x = 5y - 56,$
$2x = 7y.$

15. $3\frac{1}{8}x = \frac{1}{8}y + 1,$
$x - y = 2\frac{2}{5}.$

16. $x + 5y = 40.$
$x - 7y = -32.$

SUMMARY

I. **Definitions.**

1. Equations that express different relations between the same unknowns are called *independent equations.* Sec. 162.

2. Two or more equations are said to be *simultaneous* when all of them are satisfied by the same values of the unknowns. Sec. 163.

3. Two or more equations considered together are called a *system of equations.* Sec. 164.

4. Equations in which the unknowns are involved to the first degree only are called *equations of the first degree.* Sec. 165.

II. **Processes.**

1. To solve equations of the first degree in one unknown quantity, apply the properties of the equation. Sec. 158.

2. To solve independent simultaneous equations of the first degree in two unknowns, apply the method of *Addition and Subtraction* or the method of *Substitution.*

3. The method of addition and subtraction consists in multiplying one or both of the given equations by such numbers that the coefficients of one of the unknowns become equal. Then by subtraction this unknown is eliminated, and the solution is reduced to that of a single equation. Sec. 167.

4. If the coefficients of one unknown are made numerically equal, but have opposite signs, the equations should be added. Sec. 167.

5. The method of substitution consists in expressing one unknown in terms of the other by means of one equation and substituting this value in the other equation, thus eliminating one of the unknowns. Sec. 169.

REVIEW

WRITTEN EXERCISES

Solve and test:

1. $5800 + x = 20\,x.$
2. $x + 34 = 4(x + 4).$
3. $41 + x = 3(5 + x).$
4. $(x + 1)^2 - x^2 = 25.$
5. $x + x + \frac{x}{5} + x + \frac{5\,x}{12} = 5425.$
6. $x + y = 29,$
   $2\,x + 5\,y = 103.$
7. $x + y = 480,$
   $12\,x + 20\,y = 7520.$
8. $x = 4\,y + 76,$
   $x - y = 430.$

9. If a tennis ball rebounds to $\frac{3}{4}$ of the height from which it was dropped, from what height must it be dropped to rebound $3\frac{3}{4}$ ft.?

10. How high will the same ball rise on the second rebound? On the third?

11. A tower and flag staff are together 100 ft. high; the staff is $\frac{1}{4}$ of the height of the tower; find the height of each.

12. A house and lot are worth \$3500; the house is worth 6 times as much as the lot; find the value of each.

13. The part of a bridge pier under water is $\frac{2}{3}$ as high as the part out of water; the whole height is 45 ft.; find the height of each part.

14. The area of the United States is 3,025,600 sq. mi.; if this area were diminished by 600 sq. mi., the result would be 25 times that of the British Isles. What is their area?

15. Japan's annual exports in a recent year were valued at approximately \$150,000,000; this was a gain of 400% on their value ten years before. What was their value at that time?

16. It is estimated that the part of the population of the United States living on farms is $\frac{2}{3}$ of the rest of the population. Taking the total to be 80 millions, how many live on farms?

17. The average creamery of the Eastern states produces only $\frac{3}{5}$ as much butter as the average creamery of the Western states; two average Eastern creameries and three average Western creameries together produce 32,000 lb. annually. What is the annual output of each?

18. The whole amount of cheese produced annually in the United States is about 280,000,000 lb.; the amount produced by Wisconsin and New York is $\frac{3}{4}$ of that produced by the rest of the country. How many pounds are produced by these two states?

19. If a certain vat held 1000 lb. of milk more, it would hold 3000 qt. Taking the weight of 1 qt. of milk to be 2 lb., what is the vat's capacity in pounds?

20. There are 60 lb. in a bushel of wheat and 196 lb. of flour in a barrel. Wheat loses 18% by weight in being ground into flour. How many bushels of wheat are required to make a barrel of flour?

21. The income from a certain investment is devoted to scholarships of two grades; the higher grade receives $200 more than the lower per scholarship. When there are 7 students holding the lower grade and 7 holding the higher, the income exceeds the expense by $100; but the income is exactly sufficient to provide 8 scholarships of the higher grade and 5 of the lower grade. Find the amount of the income, and the amount of each grade of scholarships.

CHAPTER XI

FRACTIONS

PRELIMINARY DEFINITIONS AND LAWS

170. Meaning of Fraction. In arithmetic the fraction $\frac{4}{12}$ is taken to mean either 4 of the 12 equal parts of a unit, the quotient of $4 \div 12$, or the ratio of 4 to 12.

All three questions:

What part of 12 *is* 4?
What is the quotient of $4 \div 12$?
What is the ratio of 4 *to* 12?

are answered by one fraction, $\frac{4}{12}$.

171. Fractions in Algebra. In algebra, similarly, the symbol $\frac{a}{b}$ stands for a of the b equal parts of a unit, or for the quotient of $a \div b$, or for the ratio of a to b; but it is usually regarded as an indicated division.

Symbols like $\frac{a}{b}$, $\frac{19}{275}$, and $\frac{a+x}{4q+p^2}$ are therefore usually read "a divided by b," "19 divided by 275," and "$a+x$ divided by $4q+p^2$." For brevity, they may be read "a over b," "19 over 275," and the like.

172. The dividend and the divisor of the indicated division are called the **numerator** and the **denominator** of the fraction; together they are called the **terms** of the fraction.

173. A fraction is said to be in its **lowest terms** when its numerator and denominator have no common factor.

ORAL EXERCISES

1. State three meanings for $\frac{2}{5}$. For $\frac{9}{7}$. For $\frac{a}{b}$. For $\frac{a+b}{c}$.

2. What is the numerator of the fraction that represents the ratio of a to $b+c$? What fraction represents the ratio of $c+d$ to $c-d$?

3. What fraction denotes the division of m by pq?

WRITTEN EXERCISES

1. A traveler spends a dollars the first week, b dollars the second week, and c dollars the third. What fraction expresses his average expense per day? If a, b, and c are all equal, what is the daily average?

2. The ages of three students are a, $a+2$, $a+6$ years respectively. What fraction represents their average age? If $a=15$, what is their average age?

3. A passenger train running p mi. per hour and a freight train running f mi. per hour in the opposite direction leave the same place at the same time. What represents their distance apart in one hour? What fraction expresses the number of hours before they will be d miles apart?

4. A passenger train running p mi. per hour is behind a freight train running f mi. per hour in the same direction. How much does the faster train gain on the slower per hour?

5. According to Exercise 4, when the passenger train is d miles behind the other, what fraction represents the number of hours before it will overtake the other?

174. The Sign of a Fraction. Every fraction, taken as a whole, has a sign before it, expressed or understood, in addition to the signs that the numerator and the denominator may contain.

175. The Law of Signs in Fractions. The sign of the quotient is changed if the sign of either the divisor or the dividend is changed (Sec. 151, p. 99); hence,

To change the sign of either the numerator or the denominator is equivalent to changing the sign of the fraction.

Thus, if $\frac{2}{3}$ is changed to $\frac{-2}{3}$, the latter is the same as $-\frac{2}{3}$.

Similarly, if $\frac{2}{3}$ is changed to $\frac{2}{-3}$, the latter is the same as $-\frac{2}{3}$.

If the signs of both numerator and denominator are changed, the value of the fraction is unchanged.

Thus, $\frac{-a}{-b}=\frac{a}{b}$, also $\frac{b(-a)}{d(-c)}=\frac{-b(-a)}{-d(-c)}=\frac{a\cdot b}{c\cdot d}$.

ORAL EXERCISES

State expressions equal to these, and having no negative signs in the numerator or denominator of the fractions:

1. $\frac{-3}{5}$.
2. $\frac{7}{-9}$.
3. $\frac{-4}{-7}$.
4. $\frac{-a}{x}$.
5. $\frac{-3a}{-2y}$.
6. $\frac{4}{-d}$.
7. $\frac{-gt}{3}$.
8. $\frac{1}{-am}$.
9. $\frac{-x}{-3y}$.
10. $\frac{(-3)(-4)}{5}$.
11. $\frac{(-a)(-b)}{c}$.
12. $\frac{(-2)(3x)}{(-a)(-b)}$.
13. $\frac{4}{(-3)(-5)}$.
14. $\frac{-4}{(-3)5}$.
15. $\frac{-m}{p(-q)}$.
16. $\frac{2(-3x)}{a(-b)}$.
17. $\frac{3a(-2b)}{5t}$.
18. $\frac{4h}{(-3)(-5x)}$.
19. $\frac{(-2)(-3x)}{(-a)(-b)}$.
20. $\frac{-8x}{5(-3y)}$.

Express with denominator $x-y$:

21. $\frac{-7}{y-x}$.
22. $\frac{(-3)(-m)}{y-x}$.
23. $\frac{-a}{y-x}$.
24. $\frac{(-2a)(-5b)}{y-x}$.
25. $\frac{a-b}{y-x}$.
26. $\frac{36(a-b)}{y-x}$.

Express with numerator $m-r$:

27. $\frac{r-m}{2}$.
28. $\frac{r-m}{2x+3y}$.
29. $\frac{r-m}{-3q}$.
30. $\frac{r-m}{(-5x)(3y)}$.
31. $\frac{r-m}{a-b}$.
32. $\frac{r-m}{-3(x-a)}$.

33. Are the fractions $\frac{a+b}{a-b}$, $\frac{a+b}{b-a}$ equal? How is one related to the other?

Compare similarly:

34. $\frac{b-a}{-2}$ and $\frac{a-b}{2}$; $\frac{b-a}{-c-d}$ and $\frac{a-b}{c+d}$.

35. $\frac{a(-b)(-c)}{ef}$ and $\frac{abc}{ef}$; $\frac{(-a)(-b)(-c)}{ef}$ and $\frac{abc}{ef}$.

36. $\frac{a(-b)(-c)}{-ef}$ and $\frac{abc}{ef}$; $\frac{a(-b)(-c)}{(-e)(-f)}$ and $\frac{abc}{ef}$.

WRITTEN EXERCISES

For each of the following write an equal fraction preceded by the sign + and having the same denominator as the original fraction:

1. $-\frac{-x}{y}$. 5. $-\frac{5x}{3q}$. 9. $-\frac{1}{a}$. 13. $-\frac{a-b}{2m}$.

2. $-\frac{x}{-y}$. 6. $-\frac{-3m}{8}$. 10. $-\frac{-1}{b}$. 14. $-\frac{a-b}{a+b}$.

3. $-\frac{-a}{-2b}$. 7. $-\frac{2d}{9c}$. 11. $-\frac{1}{-b}$. 15. $-\frac{4x-3}{2a-1}$.

4. $-\frac{4a}{7b}$. 8. $-\frac{1}{3}$. 12. $-\frac{a+b}{3x}$. 16. $-\frac{2a-5}{3-2a}$.

17. $-\frac{3a+2b}{4x+5}$. 18. $-\frac{5m-1}{3+4x}$. 19. $-\frac{2-x^2}{x^2-3}$. 20. $-\frac{t^3+5}{1-t}$.

21–40. For each of the fractions in Exercises 1–20 write an equal fraction preceded by the sign + and having the same numerator as the given fraction.

Find the value of each of the following fractions when $a=-\frac{3}{4}$ and $b=-\frac{1}{2}$:

41. $\frac{-a}{b}$. 42. $\frac{-2a}{-4b}$. 43. $\frac{4a}{-2b}$. 44. $\frac{a+b}{a-b}$.

Find the value of each of the following fractions when $x = -2$ and $y = 3$:

45. $\dfrac{2x}{-y}$. 47. $\dfrac{x-y}{x+y}$. 49. $\dfrac{2xy}{y-x}$. 51. $\dfrac{x^2y^2}{x^2+y^2}$.

46. $\dfrac{\frac{1}{2}x}{4y}$. 48. $\dfrac{x-y}{xy}$. 50. $\dfrac{x^2-y^2}{xy}$. 52. $\dfrac{-2x-y}{x-2y}$.

176. If the numerator of a fraction is of the same degree as the denominator, or of higher degree, the fraction may be reduced to an integral expression or to an integral expression and a fraction (mixed expression).

To do this, *divide the numerator of the given fraction by the denominator.*

For example:

$$\frac{x^2+1}{x^2+x+1} = 1 - \frac{x}{x^2+x+1}, \text{ a mixed expression,}$$

$$\frac{20x^2-5x+3}{10x} = \frac{20x^2}{10x} - \frac{5x}{10x} + \frac{3}{10x} = 2x - \frac{1}{2} + \frac{3}{10x},$$

$$\frac{x^3+x^2+1}{2x+3} = \frac{1}{2}x^2 - \frac{1}{4}x + \frac{3}{8} - \frac{1}{8(2x+3)},$$

and $\dfrac{a^3-b^3}{a^2+ab+b^2} = a - b$, an integral expression.

WRITTEN EXERCISES

Reduce to integral or mixed form:

1. $\dfrac{at+\frac{1}{2}at^2}{a}$. 6. $\dfrac{s(p+u)^2}{pu}$. 11. $\dfrac{a^2-3a}{a-2}$.

2. $\dfrac{w-w^2}{w}$. 7. $\dfrac{25a^2b^2}{5ab}$. 12. $\dfrac{36xy+5}{9x}$.

3. $\dfrac{2kt-k}{k}$. 8. $\dfrac{375x^3y^2}{25x^2y}$. 13. $\dfrac{2a^3+3a^2+1}{a}$.

4. $p\left(\dfrac{100+a}{100}\right)$. 9. $\dfrac{a^2+b^2}{a+b}$. 14. $\dfrac{x^2+y+3}{x^2}$.

5. $\dfrac{(t-tr)^2}{t^2}$. 10. $\dfrac{x^2+a^2}{x-a}$. 15. $\dfrac{4x^4+10x^2+5}{2x^2}$.

177. Mixed expressions are changed to the fractional form by reversing the process of Section 176. That is,

Multiply the integral part by the denominator of the fraction and add the product to the numerator of the fraction. The sum is the numerator of the result.

For example:

$$x^2 + x + \frac{2}{x} = \frac{x(x^2 + x) + 2}{x} = \frac{x^3 + x^2 + 2}{x}.$$

WRITTEN EXERCISES

Change to fractional forms:

1. $a + 2b + \frac{c}{d}$.
2. $x + \frac{2y}{3z}$.
3. $a^2 + \frac{3b}{5a^2}$.
4. $x^2 + 2xy + \frac{3}{y}$.
5. $17x^4 + 3x^3 + \frac{1}{x^2}$.
6. $a^2 + 2ab + b^2 + \frac{c}{b}$.
7. $x^3 + x^2 + \frac{1}{x+1}$.
8. $x^2 + 2 - \frac{1}{x^2 - 2}$.
9. $p^2 - \frac{su}{pv}$.
10. $1 - \frac{w^2}{w}$.
11. $p + \frac{pr}{100}$.
12. $vt + \frac{1}{v}$.

178. Principle of Reduction. *The value of a fraction is unaltered if both numerator and denominator are multiplied or divided by the same number.*

This property follows from Sec. 151, p. 98.

For example:

$$\frac{6}{2} = 3 \text{ and } \frac{2 \cdot 6}{2 \cdot 2} = \frac{12}{4} = 3.$$

$$\frac{24}{6} = 4 \text{ and } \frac{24 \div 3}{6 \div 3} = \frac{8}{2} = 4.$$

179. To Reduce Fractions to Lowest Terms, *divide both numerator and denominator by all factors common to them.*

The division may be indicated by canceling.

For example, $\frac{x\cancel{(a-x)}}{2y\cancel{(a-x)}}$ is reduced to $\frac{x}{2y}$ by dividing both numerator and denominator by $a - x$, their only common factor.

ORAL EXERCISES

Reduce to lowest terms:

1. $\frac{8}{12}$.
2. $\frac{5}{10}$.
3. $\frac{14}{28}$.
4. $\dfrac{ab}{ac}$.
5. $\dfrac{ab}{ac^2}$.
6. $\dfrac{18\,x^2y^2}{12\,x^2}$.
7. $\dfrac{mnp}{mnq}$.
8. $\dfrac{a^2x}{a^2y}$.
9. $\dfrac{abm}{abp}$.
10. $\dfrac{ab^2c}{ad^2c}$.
11. $\dfrac{abc}{amn}$.
12. $\dfrac{x(x+y)}{2(x+y)}$.

180. Law of Exponents. The law of exponents in division (Sec. 148, p. 97) applies to fractions.

For example:

$$\frac{a^2}{a^5}=\frac{\cancel{a}\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot a\cdot a\cdot a}=\frac{\cancel{a^2}}{\cancel{a^2}\cdot a^3}=\frac{1}{a^3}\text{; also }\frac{a^4b}{a^7c}=\frac{\cancel{a^4}b}{\cancel{a^4}a^3c}=\frac{b}{a^3c}.$$

$$\frac{x^4}{x^3}=\frac{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}\cdot x}{\cancel{x}\cdot\cancel{x}\cdot\cancel{x}}=\frac{\cancel{x^3}\cdot x}{\cancel{x^3}}=x\text{; also }\frac{14\,a^3b^2c}{7\,a^2bc}=\frac{2\cdot\cancel{7}\,\cancel{a^2}\cancel{b}\cancel{c}ab}{\cancel{7}\,\cancel{a^2}\cancel{b}\cancel{c}}=2\,ab.$$

$\dfrac{a^m}{a^r}$ means m factors, each a, in the numerator, and r factors, each a, in the denominator. They can be canceled from both numerator and denominator, one by one, until they are exhausted in either the numerator or the denominator.

ORAL EXERCISES

Reduce to lowest terms, and express without negative signs in the numerator or the denominator of the fraction itself:

1. $\dfrac{a^2}{a^5}$.
2. $\dfrac{3\,b^5}{4\,b^9}$.
3. $\dfrac{-x^7}{x^{13}}$.
4. $\dfrac{m^5}{m^2}$.
5. $\dfrac{-y^4}{2\,y^3}$.
6. $\dfrac{-t^5}{-t^8}$.
7. $\dfrac{-12\,p^3}{16\,p^5}$.
8. $\dfrac{20\,r^3}{-15\,r}$.
9. $-\dfrac{3\,c^3}{-2\,c^4}$.
10. $-\dfrac{-a}{-a^4}$.
11. $-\dfrac{-g^7}{4\,g^3}$.
12. $-\dfrac{3\,a^2}{15\,a^8}$.
13. $\dfrac{-x^{7n}}{x^{10n}}$.
14. $\dfrac{a^{4n}}{a^{10n}}$.
15. $\dfrac{10^8}{10^{11}}$.
16. $\dfrac{2^{4a}}{-2^{7a}}$.

WRITTEN EXERCISES

Reduce to lowest terms and express without using negative signs in the numerator or denominator of the fraction:

1. $\dfrac{-a^2b}{-ab^2}$.
2. $\dfrac{-a^2bc}{a^2c^2}$.
3. $-\dfrac{3ab^2c^2}{-6a^3bc^3}$.
4. $\dfrac{2^7a^{3p}}{2^4a^{2p}}$.
5. $\dfrac{mqv^2}{2q}$.
6. $\dfrac{\frac{1}{2}mv^2}{m}$.
7. $\dfrac{mas}{ma}$.
8. $\dfrac{-a^{3n}b^{7q}}{a^{6n}b^{5q}}$.
9. $\dfrac{mat}{m}$.
10. $\dfrac{-12x^2y^2z^{10}}{24x^4y^3z^5}$.
11. $\dfrac{a(-b^2)(-c^3)}{3a^2b(-c^2)}$.
12. $\dfrac{x^{n-1}y^{2n+2}z^p}{x^{n-1}y^{n+2}z^{2p}}$.

181. Sometimes the common factors are made more apparent by factoring either the numerator or the denominator, or both.

For example:

$$\frac{ax-ay}{cx-cy}=\frac{a(x-y)}{c(x-y)}=\frac{a}{c}.$$

$$\frac{a^2-m^2}{4(a-m)}=\frac{(a+m)(a-m)}{4(a-m)}=\frac{a+m}{4}.$$

$$\frac{-5x-10a}{x^2+4ax+4a^2}=\frac{-5(x+2a)}{(x+2a)^2}=\frac{-5}{x+2a}=-\frac{5}{x+2a}.$$

WRITTEN EXERCISES

Reduce to lowest terms:

1. $\dfrac{a^2-b^2}{b-a}$.
2. $\dfrac{3x-3y}{7x-7y}$.
3. $\dfrac{ab-ac}{3b-3c}$.
4. $\dfrac{a^2-x^2}{4a+4x}$.
5. $\dfrac{1-y^2}{a+ay}$.
6. $\dfrac{a-b}{b-a}$.
7. $\dfrac{a-t}{a^2-t^2}$.
8. $\dfrac{a^2+ax}{ab+bx}$.
9. $\dfrac{x^2-2x+1}{x^2-x}$.
10. $\dfrac{a^2+2ap+p^2}{ab+bp}$.
11. $\dfrac{x^2-y^2}{y-x}$.
12. $\dfrac{d^2-9}{d^2+6d+9}$.
13. $\dfrac{(a^2-x^2)z}{(a-x)2y}$.
14. $\dfrac{a^2+b^2}{a^4-b^4}$.
15. $\dfrac{m^2-2mr+r^2}{m^3-r^3}$.

182. *The same factor may be introduced into both numerator and denominator of a fraction without altering its value.* Sec. 178.

For example:

$$\frac{5}{7}=\frac{3\cdot 5}{3\cdot 7}=\frac{15}{21};\ \text{also}\ \frac{-m}{r}=\frac{(-1)(-m)}{(-1)r}=\frac{m}{-r}.$$

$$\frac{a}{x}=\frac{a^2\cdot a}{a^2\cdot x}=\frac{a^3}{a^2x};\ \text{also}\ \frac{a+x}{a-x}=\frac{(a+x)(a+x)}{(a+x)(a-x)}=\frac{(a+x)^2}{a^2-x^2}.$$

ORAL EXERCISES

For each of the following, state an equal fraction having the denominator a^2b:

1. $\frac{4}{ab}$. 2. $\frac{-5}{a^2}$. 3. $\frac{8}{-b}$. 4. $\frac{-x}{a}$. 5. $\frac{2y}{1}$.

State an equal fraction having the denominator $12a^2t^2$.

6. $\frac{b}{12a^2}$. 8. $\frac{x}{a^2}$. 10. $\frac{-1}{1}$. 12. $\frac{2}{-a^2t^2}$. 14. $\frac{c}{4at}$.

7. $\frac{-5}{6at^2}$. 9. $\frac{-t}{1}$. 11. $\frac{-mx}{4}$. 13. $\frac{a}{12t^2}$. 15. $\frac{-1}{3a^2t}$.

WRITTEN EXERCISES

Write equal fractions having the denominator $a(b^2-x^2)$:

1. $\frac{3}{b^2-x^2}$. 2. $\frac{a-x}{b+x}$. 3. $\frac{a(b+x)}{b-x}$. 4. $\frac{b-x}{a}$. 5. $\frac{-3}{a(x-b)}$.

183. When several fractions have the same denominator, that denominator is called their **common denominator**. The common denominator must evidently be a multiple of the given denominators. When it is their l.c.m. it is called the **lowest common denominator** (l.c.d.) of the given fractions.

184. *To reduce fractions to their lowest common denominator, find the l. c. m. of their denominators and multiply the numerator and denominator of each fraction by the quotient of its denominator and the l. c. m.*

ORAL EXERCISES

Reduce to lowest common denominator:

1. $\frac{2}{3}, \frac{5}{6}$.
2. $\frac{1}{3a}, \frac{5}{2a}$.
3. $\frac{b}{4a}, \frac{c}{6a^2}$.
4. $\frac{4}{3}, \frac{-a}{6b}$.
5. $\frac{-1}{2x}, \frac{x}{2}$.
6. $\frac{2x}{3a^2}, \frac{-5b}{6ac}$.
7. $\frac{-1}{a}, \frac{b}{a^3}$.
8. $\frac{b}{a}, \frac{a}{b}$.
9. $\frac{-1}{3a}, \frac{a}{2b}$.

WRITTEN EXERCISES

Change to fractions having the l.c.d.:

1. $\frac{2}{3x}, \frac{3}{4x^2}, \frac{1}{6x^3}$.
2. $\frac{a}{b}, \frac{b}{3c}, \frac{c}{2d}$.
3. $\frac{a}{2bx}, \frac{c}{abxy}, \frac{b}{3acx}$.
4. $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$.
5. $\frac{x^2}{2ab}, \frac{y^2}{3ac}, \frac{z^2}{4bc}$.
6. $\frac{a}{1-x}, \frac{1}{1-x^2}$.
7. $\frac{c}{a-c}, \frac{a}{a+c}$.
8. $\frac{4}{ax+x^2}, \frac{2}{ax-x^2}$.
9. $\frac{a+x}{a-x}, \frac{a-x}{a+x}$.
10. $\frac{4x^2}{3(a+b)}, \frac{xy}{6(a^2-b^2)}$.
11. $\frac{x^2}{a^2+b^2}, \frac{y^2}{a^2-b^2}$.
12. $\frac{1}{a-b}, \frac{1}{a+b}, \frac{abc}{a^2-b^2}$.
13. $\frac{3}{8(1-x)}, \frac{1}{8(1+x)}, \frac{x-1}{4(1+x^2)}$.
14. $\frac{1}{4a^3(a+b)}, \frac{1}{4a^3(a-b)}, \frac{1}{2a^2(a^2-b^2)}$.

ADDITION AND SUBTRACTION OF FRACTIONS

185. PREPARATORY.

1. What is the sum of $\frac{3}{8}$ and $\frac{1}{8}$? Of $\frac{1}{5}$ and $\frac{4}{5}$?

2. How must fractions be expressed before being added or subtracted?

3. What is the sum of $\frac{a}{b}$ and $\frac{c}{b}$? Of $\frac{x}{abc}$ and $\frac{y^2}{abc}$?

4. Subtract $\frac{ax}{bc}$ from $\frac{ay}{bc}$. Also, $\frac{m^2}{pq}$ from $\frac{n^2}{pq}$.

186. To Add or Subtract Fractions. 1. *Find the l. c. d. of the given fractions. This is the denominator of the result.*

2. *Reduce the given fractions to fractions having the l. c. d.*

3. *Find the algebraic sum of the numerators of the fractions resulting from step 2. This is the numerator of the result.*

EXAMPLE

Add $\frac{a}{a-x}$ and $\frac{3a}{a+x}$.

1. The l. c. d. is $(a-x)(a+x)=a^2-x^2$.

2. $$\frac{a}{a-x}=\frac{a(a+x)}{a^2-x^2}=\frac{a^2+ax}{a^2-x^2}.$$

3. $$\frac{3a}{a+x}=\frac{3a(a-x)}{a^2-x^2}=\frac{3a^2-3ax}{a^2-x^2}.$$

4. $$\therefore \frac{a}{a-x}+\frac{3a}{a-x}=\frac{a^2+ax}{a^2-x^2}+\frac{3a^2-3ax}{a^2-x^2}=\frac{4a^2-2ax}{a^2-x^2}.$$

ORAL EXERCISES

Add:

1. $\frac{7}{13}, \frac{4}{13}$.
2. $\frac{7}{a}, \frac{4}{a}$.
3. $\frac{5}{2}, \frac{9}{4}$.
4. $\frac{5}{2a}, \frac{9}{4a}$.
5. $\frac{5b}{3c}, \frac{9b}{2c}$.
6. $\frac{4c}{3x}, \frac{5c}{12x}$.

WRITTEN EXERCISES

Add:

1. $\frac{a}{ax}, \frac{b}{x}$.
2. $\frac{a}{bc}, \frac{c}{xy}$.
3. $\frac{m}{n}, \frac{p}{q}$.
4. $\frac{3p}{4q}, \frac{p}{2q}$.
5. $\frac{1}{mn}, \frac{1}{pq}$.
6. $\frac{1}{6z}, \frac{5}{12z}$.
7. $\frac{a}{x}, \frac{b}{y}$.
8. $\frac{1}{x^2}, \frac{5}{xy}$.
9. $\frac{b}{a}, \frac{w}{p}$.
10. $\frac{3}{a}, \frac{5}{a+1}$.
11. $\frac{4}{1-x}, \frac{5}{1+x}$.
12. $\frac{2t-4}{(t+3)^2}, \frac{1}{t+3}$.
13. $\frac{c}{a}, \frac{c+x}{2a}$.
14. $\frac{a+x}{a-x}, \frac{3x+5a}{2(a-x)}$.
15. $\frac{1-p}{3v+2}, \frac{4p+5}{9v+6}$.

Subtract the second fraction from the first:

16. $\frac{3}{b}, \frac{2}{b}$.

17. $\frac{3a}{b}, \frac{2b}{a}$.

18. $\frac{1}{x}, \frac{1}{3x}$.

19. $\frac{5}{x}, \frac{x^2}{x^3}$.

20. $\frac{7}{a}, \frac{4}{ab}$.

21. $\frac{a}{b}, \frac{c}{d}$.

22. $\frac{4x}{a+5}, \frac{2x}{3(a+5)}$.

23. $\frac{7}{1-x}, \frac{5}{(1-x)^2}$.

24. $\frac{8}{m+1}, \frac{2}{m-1}$.

187. Signs before Fractions. Since the sign before the fraction relates to the fraction as a whole, the *whole* numerator is added or subtracted, as the case may be, the bar of the fraction having upon the numerator the effect of a parenthesis.

For example:

$$\frac{a}{c}-\frac{d-e}{c}=\frac{a-(d-e)}{c}=\frac{a-d+e}{c}.$$

$$\frac{a}{c}-\frac{d-e+f}{c}=\frac{a-(d-e+f)}{c}=\frac{a-d+e-f}{c}.$$

$$\begin{aligned}\frac{a}{x+1}+\frac{2a}{x-1}-\frac{5a-7ax}{x^2-1}&=\frac{ax-a}{x^2-1}+\frac{2ax+2a}{x^2-1}-\frac{5a-7ax}{x^2-1}\\&=\frac{ax-a+(2ax+2a)-(5a-7ax)}{x^2-1}\\&=\frac{ax-a+2ax+2a-5a+7ax}{x^2-1}\\&=\frac{10ax-4a}{x^2-1}.\end{aligned}$$

WRITTEN EXERCISES

Perform the operations indicated:

1. $\frac{5}{x}-\frac{4}{7x}-5x$.

2. $\frac{x-xy}{4}-\frac{x-xy}{2}$.

3. $\frac{x+y}{z}+\frac{x-2y}{2z}$.

4. $\frac{1}{x}-\frac{3-x}{x^2}$.

5. $\frac{x-2y}{5}+\frac{2x-y}{3}$.

6. $\frac{z}{2m^2q}+\frac{w}{2mq^2}$.

7. $\frac{3a^2b}{5x^2y}-\frac{5ab^2}{3xy}$.

8. $\frac{x}{ab}+\frac{y}{bc}4xbc$.

9. $\frac{a}{c}-\frac{c}{d}-3ad+\frac{1}{2}c^2$.

10. $\frac{3a}{5x}-\frac{6b}{10y}+15xy$.

11. $\dfrac{a^2}{2xy}+\dfrac{4b^2-3a^2}{x^2}$.

12. $\dfrac{1}{a-b}+\dfrac{1}{a+b}$.

13. $\dfrac{a}{3}+\dfrac{a}{5}-\dfrac{a}{15}-5a$.

14. $\dfrac{a}{b^3c^2}+\dfrac{3b}{a^2c^3}+\dfrac{1}{ab^2c^2}$.

15. $\dfrac{a}{b}-\dfrac{b}{a}+\dfrac{b^2}{ab}+a^2b$.

16. $\dfrac{x^6}{x^4-a^4}-\dfrac{a^4x^2}{a^4+x^4}$.

17. $\dfrac{m}{q}+n-\dfrac{2n}{3q}$.

18. $\dfrac{x}{y}-\dfrac{w}{xy}-\dfrac{z}{wy}$.

19. $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{d}$.

20. $a+\dfrac{c}{6abxy}+\dfrac{b}{3acx}$.

21. $\dfrac{x-y}{x^2y}+\dfrac{x-y}{xy^2}$.

22. $\dfrac{a}{2b}-\dfrac{a-b}{2(a+b)}$.

23. $\dfrac{x}{2y}+\dfrac{x+y}{3(x-y)}$.

24. $\dfrac{a^2+b^2}{ab}-\dfrac{b^2+c^2}{bc}$.

25. $\dfrac{a+b}{a}+\dfrac{b+c}{b}+\dfrac{c+a}{c}$.

26. $\dfrac{a-b}{ab}+\dfrac{c-a}{ac}+\dfrac{b-c}{bc}$.

27. $\dfrac{1}{a-x}-3+\dfrac{2a}{(a+x)^2}$.

28. $\dfrac{1}{(x^2+1)^2}+\dfrac{x-1}{x^2+1}$.

29. $\dfrac{1}{2(x-1)}-\dfrac{1}{3(x+1)}-\dfrac{1}{x^2}$.

30. $\dfrac{2x^2-y^2}{x^2}-\dfrac{y^2-z^2}{y^2}-\dfrac{z^2-x^2}{z^2}$.

31. $\dfrac{1}{2(a-b)}+\dfrac{1}{2(a+b)}+\dfrac{a}{a^2+b^2}$.

32. $2-\dfrac{x^2-y^2}{x^2+y^2}+\dfrac{x^2+y^2}{x^2-y^2}$.

33. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{a^2-b^2}$.

34. $\dfrac{x}{x^2-y^2}+\dfrac{y}{(x+y)^2}$.

35. $\dfrac{a}{(b-c)(b-a)}+\dfrac{b}{(c-a)(c-b)}+\dfrac{c}{(a-b)(a-c)}$.

188. Preparatory.

Read as the sum or difference of two fractions:

1. $\dfrac{x+y}{3}$.
2. $\dfrac{3x+5y}{4}$.
3. $\dfrac{a+5}{b}$.
4. $\dfrac{6+x}{5}$.
5. $\dfrac{x-5}{a}$.
6. $\dfrac{3-a}{7b}$.
7. $\dfrac{4a+x}{a+9}$.
8. $\dfrac{x+1}{x-1}$.

189. Separating Fractions into Parts. It is sometimes desirable to separate fractions into two or more addends, by reversing the process used in the addition of fractions.

For example:

1. $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}.$

2. $\frac{ax+5y}{10ay}=\frac{ax}{10ay}+\frac{5y}{10ay}=\frac{x}{10y}+\frac{1}{2a}.$

3. $$\begin{aligned}\frac{5ay-4bc^2}{12ab}-\frac{7cy+16b}{12bc}&=\frac{5ay}{12ab}-\frac{4bc^2}{12ab}-\frac{7cy}{12bc}-\frac{16b}{12bc}\\&=\frac{5y}{12b}-\frac{c^2}{3a}-\frac{7y}{12b}-\frac{4}{3c}\\&=-\frac{y}{6b}-\frac{c^2}{3a}-\frac{4}{3c}\\&=-\frac{1}{3}\left(\frac{y}{2b}+\frac{c^2}{a}+\frac{4}{c}\right).\end{aligned}$$

4. $$\begin{aligned}\frac{(a+x)^2-3(a-x)}{a^2-x^2}=\frac{(a+x)^2}{a^2-x^2}-\frac{3(a-x)}{a^2-x^2}&=\frac{a+x}{a-x}-\frac{3}{a+x}\\&=\frac{a}{a-x}+\frac{x}{a-x}-\frac{3}{a+x}.\end{aligned}$$

WRITTEN EXERCISES

Write each fraction as the algebraic sum of two or more fractions, reduce the fractions to lowest terms, and unite those with the same denominator:

1. $\frac{5x+8y}{10a}.$

2. $\frac{6a^2-3b^2}{14ab}.$

3. $\frac{a+7b+c}{21a^2x}.$

4. $\frac{a^2x+4b^2y}{12ay}.$

5. $\frac{a+x}{y}+\frac{a^2b-5dx}{aby}.$

6. $\frac{3g+5h}{30(g+1)}.$

7. $\frac{9y-14x}{6xy}+\frac{15z-2ax}{6xz}-xyz.$

8. $\frac{7y-3x}{3y}-\frac{4ay-9ax+15y}{3ay}+6a^2y^2.$

9. $\frac{ab^2+8b+8a}{8ab}-\frac{6bc-18ac-6ab}{6abc}.$

10. $\dfrac{(a-1)^2-11(a+1)}{a^2-1}$. 11. $\dfrac{a^2-b^2+5}{a^2-2ab+b^2}$.

12. $\dfrac{4(1+r)+3r(1-r)}{1-r^2}-\dfrac{7a(1+r)-6r(1-r)}{a-ar^2}$.

MULTIPLICATION OF FRACTIONS

190. Multiplying Fractions. *The product of two or more fractions is the product of their numerators divided by the product of their denominators.*

For example:

$$\frac{2}{3}\cdot\frac{5}{7}=\frac{2\cdot 5}{3\cdot 7}=\frac{10}{21}.$$

$$\frac{m}{n}\cdot\frac{p}{q}=\frac{mp}{nq},\text{ and }\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{3e}{4}=\frac{3ace}{4bd}.$$

$$\frac{a}{c}\cdot\frac{-b}{d}=\frac{a\cdot -b}{c\cdot d}=\frac{-ab}{cd}=-\frac{ab}{cd}.$$

191. Since every integer or integral expression can be regarded as a fraction with denominator 1, the above definition of product also includes the case where one of the factors is an integral expression.

For example:

$$3\cdot\frac{5}{6}=\frac{3}{1}\cdot\frac{5}{6}=\frac{3\cdot 5}{1\cdot 6}=\frac{15}{6}.$$

$$\frac{a}{b}\cdot c=\frac{a}{b}\cdot\frac{c}{1}=\frac{ac}{b}.$$

In simplifying the result, canceling may be helpful.

For example:

$$\frac{4a}{7b}\cdot\frac{14b}{a^2}=\frac{4\cancel{a}}{\cancel{7}\cancel{b}}\cdot\frac{\overset{2}{\cancel{14}}\cancel{b}}{\cancel{a}a}=\frac{8}{a}.$$

If necessary, factor before canceling.

For example:

$$\frac{x^2}{34}\cdot\frac{51}{ax+x^2}=\frac{x\cancel{x}}{2\cdot\cancel{17}}\cdot\frac{3\cdot\cancel{17}}{\cancel{x}(a+x)}=\frac{3x}{2(a+x)}.$$

$$\frac{x^2-4}{3+a}\cdot\frac{9+6a+a^2}{5x+10}=\frac{(x^2-4)(9+6a+a^2)}{(3+a)(5x+10)}$$

$$=\frac{(x-2)\cancel{(x+2)}(3+a)^{\cancel{2}}}{5\cancel{(3+a)}\cancel{(x+2)}}$$

$$=\frac{(x-2)(3+a)}{5}.$$

ORAL EXERCISES

Multiply:

1. $\frac{4b}{3x} \cdot \frac{1}{5}$.
2. $\frac{x^2}{5} \cdot \frac{x^3}{4}$.
3. $\frac{1}{a^3} \cdot \frac{1}{a^4}$.
4. $\frac{x^n}{3} \cdot \frac{x^n}{7}$.
5. $\frac{a}{4b} \cdot \frac{c}{3d}$.
6. $6a \cdot \frac{4b}{5c}$.
7. $p^3 \cdot \frac{p^7}{4a}$.
8. $\frac{a^n}{b^{2n}} \cdot \frac{a^5}{b^8}$.

Multiply each of the following in turn by 12, and reduce to lowest terms:

9. $\frac{8}{3ax}$.
10. $\frac{4}{2by}$.
11. $\frac{5x}{ac}$.
12. $\frac{4y}{3x}$.
13. $\frac{2m}{6aby}$.
14. $\frac{7ax}{3b}$.
15. $\frac{8d}{12c}$.
16. $\frac{5t}{24s}$.

Multiply Exercises 9–16 above in turn by:

17. $3a$. 18. $2b$. 19. xy. 20. $4ax$. 21. $3cy$.

Multiply:

22. $\frac{3}{a} \cdot \frac{a}{6}$.
23. $\frac{3}{a^2} \cdot \frac{a}{2b}$.
24. $\frac{m}{n} \cdot \frac{cn}{m^2}$.
25. $\frac{8}{x} \cdot \frac{x^3}{16}$.

WRITTEN EXERCISES

Multiply:

1. $\frac{x^5}{a^4} \cdot \frac{x^7}{a^7}$.
2. $\frac{a^2b}{cd} \cdot \frac{3a^4}{c^3d}$.
3. $\frac{4x}{5a} \cdot \frac{2y}{3ab}$.
4. $\frac{a^{5n+1}}{10^{3n+2}} \cdot \frac{a^6}{10^{2n}}$.
5. $\frac{a^2b^{n+2}}{7c^3} \cdot \frac{a^{3n}b^5}{4d^{3n}}$.
6. $\frac{2^8}{y^5} \cdot \frac{x^{3n}}{y^7z^{3n}}$.

Multiply and reduce to lowest terms:

7. $\frac{7}{13a} \cdot \frac{39a^2}{49}$.
8. $\frac{6x^2}{a} \cdot \frac{a^2}{15x}$.
9. $\frac{a}{b} \cdot \frac{cb^2}{ad}$.
10. $\frac{p}{q} \cdot \frac{q^2}{3p^3}$.
11. $\frac{a}{3c} \cdot \frac{9c}{ad}$.
12. $\frac{x}{y^2} \cdot \frac{z^2}{w}$.
13. $z^2 \cdot \frac{xy^2z}{zw}$.
14. $\frac{x}{y} \cdot \frac{x+y}{5xy}$.
15. $\frac{p^2}{q^2} \cdot \frac{aq}{bp}$.
16. $\frac{m+n}{a} \cdot ab$.
17. $\frac{1}{c^2} \cdot \frac{bc}{(x+y)^2}$.
18. $\frac{x}{y} \cdot \frac{y^2}{z^2} \cdot \frac{z^3}{w^3}$.

19. $p^4q^4 \cdot \frac{ab}{p^3q^3}$.

20. $\frac{d^2c^2}{a^2b^2} \cdot \frac{ab}{cd}$.

21. $\frac{4\,b^2}{xy^2} \cdot xy$.

22. $\frac{mn}{3\,x} \cdot 3\,mx$.

23. $\frac{3\,c}{abx} \cdot ab$.

24. $\frac{a}{bx} \cdot \frac{cx}{d}$.

25. $\frac{2\,x}{a} \cdot \frac{3\,ab}{c} \cdot \frac{3\,ac}{2\,b}$.

26. $a \cdot \frac{1}{bc} \cdot \frac{c}{a}$.

27. $3\,bc \cdot \frac{b+c}{2\,bc}$.

28. $-x^2 \cdot \frac{-x}{ab}$.

29. $\frac{1}{x} \cdot \frac{x^2-xy}{6\,bc}$.

30. $\frac{-2\,a^2}{c^2} \cdot \frac{-b}{d}$.

31 $\frac{1}{a^3b^4xy^2} \cdot \frac{1}{abx^2y}$.

32. $\frac{x^3b^2z}{ab^2c} \cdot \frac{a^3b^2c}{xy^2z^4}$.

33. $\frac{3\,m^4n^3s}{4\,xy^2z^3} \cdot \frac{6\,x^4y^3z^2}{5\,m^2ns}$.

34. $\frac{ab^2}{cd^2} \cdot \frac{c^2d}{e^2f} \cdot \frac{ef}{ab}$.

35. $(x+3) \cdot \frac{x-2}{x-3}$.

36. $\frac{4\,x}{a+5} \cdot \frac{2(a+5)}{3\,x^3}$.

37. $\frac{am-bm}{c+1} \cdot \frac{2}{ar-br}$.

38. $(a^2-1) \cdot \frac{ax}{a+1}$.

39. $\frac{x^2-2\,x+1}{q^3} \cdot \frac{3\,q^2}{x-1}$.

40. $\frac{1+6\,a+9\,a^2}{21} \cdot \frac{7\,t}{2+6\,a}$.

41. $\frac{4\,d^3}{x^2-10\,x+25} \cdot \frac{x^2-25}{8\,d}$.

42. $\frac{9-a^2}{a-4} \cdot \frac{16-8\,a+a^2}{a^2+6\,a+9}$.

43. $\frac{7\,x+14}{1-5\,a} \cdot \frac{2\,a-3}{(x+2)^2} \cdot \frac{x+2}{6\,a-9}$.

44. $\frac{a}{x-1} \cdot \frac{1-x^2}{2\,a+ay} \cdot \frac{2+y}{1+x}$.

45. $\left(\frac{1}{x}+\frac{1}{y}\right)^2$.

46. $\left(\frac{1}{m^2}+1\right)^2$.

47. $\left(\frac{x}{y}+\frac{y}{x}\right)^2$.

48. $\left(1+\frac{1}{x}\right)^2$.

49. $\left(\frac{1}{x}-1\right)^2$.

50. $\left(\frac{1}{x}-\frac{1}{y}\right)^2$.

51. $\left(\frac{x}{y}-\frac{y}{x}\right)^2$.

52. $\left(\frac{1}{m^2}-1\right)^2$.

53. $\left(\frac{1}{x}-1\right)^2$.

54. $\left(\frac{a}{b}+1\right)^2$.

55. $\left(\frac{c}{2}+a\right)^2$.

56. $\frac{1}{2}xy\left(\frac{x}{y}-\frac{y}{x}\right)$.

57. $\left(x+\frac{1}{y}\right)\left(x-\frac{1}{y}\right)$.

58. $\left(\frac{m}{n}+n\right)\left(\frac{m}{n}-n\right)$.

59. $\frac{1}{2}\,xy^2\left(\frac{1}{x}+\frac{1}{y}\right)$.

60. $-\frac{3}{4}a^2b^2\left(\frac{a^2}{b^2}-\frac{a}{b}\right)$.

61. $abc^3\left(\frac{1}{a}-\frac{1}{b}\right)$.

62. $x^2y^2\left(\frac{1}{x^2}+\frac{1}{y^2}\right)$.

192. Powers of Fractions. *Any power of a fraction equals that power of the numerator divided by the same power of the denominator.*

For example:

$$\left(-\frac{2}{3}\right)^2=\frac{(-2)^2}{(3)^2}=\frac{4}{9}.$$

$$\left(\frac{a}{b}\right)^3=\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}=\frac{a^3}{b^3}.$$

$$\left(\frac{a}{b}\right)^n=\frac{a}{b}\cdot\frac{a}{b}\cdots\text{ to } n \text{ factors.}$$

$$=\frac{a\cdot a\cdots\text{ to } n\text{ factors}}{b\cdot b\cdots\text{ to } n\text{ factors}}=\frac{a^n}{b^n}.$$

$$\left(\frac{3a^3x^2}{-5bc}\right)^3=\frac{(3a^3x^2)^3}{(-5bc)^3}=\frac{27a^9x^6}{-125b^3c^3}=-\frac{27a^9x^6}{125b^3c^3}.$$

$$\frac{a^2+6a+9}{x^2-10x+25}=\frac{(a+3)^2}{(x-5)^2}=\left(\frac{a+3}{x-5}\right)^2.$$

$$\left(\frac{m+r}{m-r}\right)^2=\frac{(m+r)^2}{(m-r)^2}=\frac{m^2+2mr+r^2}{m^2-2mr+r^2}.$$

WRITTEN EXERCISES

Write without parentheses:

1. $\left(\frac{a^3}{b^n}\right)^2.$
2. $\left(\frac{a^{4n}}{x^5}\right)^3.$
3. $\left(\frac{-2a\,3b^2}{5xy}\right)^3.$
4. $\left(1+\frac{x}{y}\right)\left(1+\frac{x}{y}\right).$
5. $\left(\frac{a}{b}\right)^{10}.$
6. $\left(\frac{a}{b}\right)^n.$
7. $\left(\frac{-3abc^2}{4m^3n}\right)^4.$
8. $\left(-\frac{2x-3}{12}\right)^3.$
9. $\left(\frac{a^2}{x^3}\right)^{3p}.$
10. $\left(\frac{1}{x^2}\right)^{q+1}.$
11. $\left(\frac{2ax}{3b^2y}\right)^5.$
12. $\left(\frac{2a^2b^3}{3x^3y}\right)^4.$
13. $\left(\frac{a}{x}-\frac{b}{y}\right)^2.$
14. $\left(\frac{-abx}{c^2dy}\right)^7.$
15. $\left(1+\frac{ab}{a+b}\right)\left(1+\frac{ab}{a+b}\right).$
16. $\left(\frac{p}{q}+\frac{q}{p}\right)\left(\frac{p}{q}+\frac{q}{p}\right).$
17. $\left(\frac{1}{a}+\frac{1}{b}\right)^2.$
18. $\left(\frac{a+bc}{a+b}\right)^2.$

Write each of the following as a power of a fraction:

19. $\dfrac{x^2-2x+1}{a^2+2a+1}$.

20. $\dfrac{a^2+2ab+b^2}{x^4-2x^3y+x^2y^2}$.

21. $\dfrac{x^2+4x+4}{4x^2+4x+1}$.

22. $\dfrac{a^2-4ab+4b^2}{9a^2-12ab+4b^2}$.

23. $\dfrac{a^6b^3}{x^3+3x^2+3x+1}$.

24. $\dfrac{100p^4-20p^2+1}{q^4-20q^2+100}$.

25. $\dfrac{a^3-9a^2+27a-27}{27x^3-27x^2+9x-1}$.

26. $\dfrac{x^2-2x+1}{x^4+2x^3y+x^2y^2}$.

DIVISION OF FRACTIONS

193. Reciprocal. If the product of two numbers is 1, each is called the **reciprocal** of the other.

Thus,

5 and $\frac{1}{5}$ are reciprocals of each other, because $5 \cdot \frac{1}{5} = 1$.

$\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals of each other, because $\frac{a}{b} \cdot \frac{b}{a} = 1$.

ORAL EXERCISES

State the reciprocal of:

1. $\frac{3}{4}$.
2. $\dfrac{1}{a}$.
3. $\dfrac{c}{d}$.
4. $\dfrac{1}{cd}$.
5. mp.
6. $\dfrac{3a}{5b}$.
7. $\dfrac{x^2}{9}$.
8. $\dfrac{14}{9ab^2}$.

194. Division of Fractions. In the division of fractions as in the division of integral expressions:

Divisor times quotient equals dividend.

Consequently, to divide $\frac{4}{5}$ by $\frac{7}{5}$ means to find a number, call it q, such that

$$\tfrac{7}{5} \text{ times } q \text{ equals } \tfrac{4}{5},$$

or,

$$\tfrac{7}{5}q = \tfrac{4}{5}.$$

To find q, we multiply both members of this equation by $\frac{5}{7}$, the reciprocal of $\frac{7}{5}$, obtaining

$$q = \tfrac{5}{7} \cdot \tfrac{4}{5}.$$

Similarly, $\frac{a}{b} \div \frac{c}{d}$ means to find a number q, such that

$$\frac{c}{d} \cdot q = \frac{a}{b}.$$

Multiplying both members by $\frac{d}{c}$, the reciprocal of $\frac{c}{d}$, we have

$$q = \frac{d}{c} \cdot \frac{a}{b}.$$

In words:

To divide by a fraction is to multiply by its reciprocal.

WRITTEN EXERCISES

Divide:

1. $\frac{1}{a} \div \frac{1}{3}$.
2. $\frac{1}{a} \div \frac{3}{a}$.
3. $\frac{1}{a} \div \frac{1}{b}$.
4. $\frac{3}{a} \div \frac{5}{b}$.
5. $\frac{a}{b} \div \frac{1}{b}$.
6. $\frac{m}{n} \div \frac{a}{b}$.
7. $\frac{a}{2b} \div \frac{c}{3d}$.
8. $\frac{m}{5n} \div \frac{5m}{21n}$.
9. $\frac{p}{7q} \div \frac{3p}{14q}$.
10. $\frac{n}{m} \div \frac{p}{q}$.
11. $\frac{x}{y} \div \frac{y}{x}$.
12. $\frac{1}{3ab} \div \frac{3}{ab}$.
13. $\frac{x^2}{y} \div \frac{x}{y^2}$.
14. $\frac{mn}{pq} \div \frac{m}{p}$.
15. $\frac{mx}{ny} \div \frac{m}{n}$.
16. $\frac{a^2}{b^2} \div \frac{a}{b}$.
17. $\frac{3a^2}{2b^2} \div \frac{2a}{3b}$.
18. $\frac{5x^2}{y^2} \div \frac{x^2}{y}$.
19. $\frac{15m^2}{n^2} \div \frac{5}{n^2}$.
20. $\frac{ab}{c} \div \frac{b}{c^2}$.
21. $\frac{6a}{b} \div \frac{a}{x}$.
22. $\frac{7a^2}{5b} \div \frac{14}{5ab^2}$.
23. $\frac{y^2z}{xw} \div \frac{yz}{x}$.
24. $\frac{m^2}{xyz} \div \frac{10m}{3x}$.
25. $\frac{a}{bx} \div \frac{d}{cx} = ?$
26. $\frac{5ax}{3cy} \div \frac{5x}{cy} = ?$
27. $\frac{b}{x-y} \div \frac{a}{x+y} = ?$
28. $\frac{3bx}{2az^q} \div \frac{ax}{2z^{b+q}} = ?$
29. $\frac{ac}{bd} \div \frac{ab}{cd} = ?$
30. $\frac{4a}{x^{m+1}} \div \frac{2b}{x} = ?$
31. $\frac{3ab}{5a^2c} \div \frac{6a^2b}{5a^3c^2} = ?$
32. $\frac{4a^2b^2}{15c^2d^2} \div \frac{4ab}{3c^2d} = ?$

33. $\frac{2x}{3y} \div \frac{4x}{3y} = ?$

34. $\frac{2x^2}{yz} \div \frac{3xyc}{z^2d} = ?$

35. $\frac{3abx}{5c^2} \div \frac{6a^2x^2}{10cb^2} = ?$

36. $\frac{2y}{3x} \div \frac{a}{b} \cdot \frac{9x^2}{4y^2} = ?$

37. $\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{e} \cdot \frac{e}{f} \div \frac{a}{f} = ?$

38. $\frac{a^{2n}b^{2n}c^{2n}}{xyz} \div \frac{a^nb^nc^n}{x^2y^2z^2} = ?$

39. $\frac{x^{2p}}{y} \cdot \frac{y^2}{x^p} \div \frac{x^3}{y^3} = ?$

40. $\frac{m^2n}{p^2q} \cdot \frac{n^2p}{q^2r} \div \frac{mnp}{qr} = ?$

41. The width of a room is f ft. How many yards wide is it? The room is to be covered with carpet $\frac{7}{8}$ of a yard wide. How many strips will be required, running the long way of the room?

42. According to Exercise 41, how many strips would be required if the carpet were $\frac{a}{b}$ of a yard wide?

43. A cook uses l pounds of sugar in making some cakes. If each cake requires $\frac{1}{2}$ of a pound, how many cakes are made? Answer the same question when $\frac{a}{b}$ of a pound is used for each cake.

44. A lamp which holds $\frac{7}{8}$ of a quart is filled from q quarts of kerosene. How many times can this be done?

45. How many times can a lamp holding $\frac{m}{n}$ of a quart be filled from q quarts of kerosene?

195. Since every integral expression can be regarded as a fraction with denominator 1, the process of division includes also the case where one of the fractions is an integral expression.

For example:

$$-\tfrac{3}{5} \div 6 = -\tfrac{3}{5} \div \tfrac{6}{1} = -\tfrac{3}{5} \cdot \tfrac{1}{6} = -\tfrac{1}{10}.$$

$$\frac{a}{-b} \div c = \frac{a}{-b} \div \frac{c}{1} = \frac{a}{-b} \cdot \frac{1}{c} = \frac{a}{-bc} \text{ or } -\frac{a}{bc}. \quad \text{Sec. 175.}$$

$$\frac{3a^2x}{by} \div (-3a) = \frac{3a^2x}{by} \cdot \frac{1}{-3a} = -\frac{ax}{by}. \quad \text{Sec. 175.}$$

$$x^2y \div \frac{-x}{z} = \frac{x^2y}{1} \cdot \frac{-z}{x} = -xyz.$$

WRITTEN EXERCISES

Perform the divisions indicated and reduce the results to lowest terms:

1. $\dfrac{3\,a^2x}{by} \div 2\,c.$

2. $\dfrac{2\,a}{b^2c} \div \dfrac{3\,a^2b}{c}.$

3. $\dfrac{-3\,a^2}{bx^2} \div a^2.$

4. $\dfrac{a^3+b^3}{a^2-b^2} \div \dfrac{a+b}{a-b}.$

5. $\dfrac{2\,ab}{3\,c^2} \div \dfrac{2\,b}{-a^3c}.$

6. $\dfrac{9\,x^2y}{12\,mn^2} \div \dfrac{3\,xy^2}{4\,m^2n}.$

7. $\dfrac{-ba^2}{7\,cb^2} \div \dfrac{3\,a}{21\,c^2b}.$

8. $\dfrac{-7\,x^3y}{122^2} \div \dfrac{7\,xy}{-4\,zw}.$

9. $\dfrac{5\,ab^2}{-21\,cd} \div \dfrac{-5\,a^2b}{7\,c^2d^2}.$

10. $\dfrac{-15\,mn^3}{25\,a^2b} \div \dfrac{3\,m^2n^2}{5\,ab}.$

11. $\dfrac{8\,a^2}{a^2b^2} \div \dfrac{4\,a}{a-b}.$

12. $\dfrac{a-x}{y} \div \dfrac{a^2-x^2}{-xy}.$

13. $\dfrac{2\,a+3\,b}{c+d} \div \dfrac{c-d}{2\,a-3\,b}.$

14. $\dfrac{a}{a+x} \div \dfrac{a^2-a}{a+x}.$

15. $\dfrac{4\,x^2-9\,y^2}{a^2-b^2} \div \dfrac{2\,x+3\,y}{a-b}.$

16. $\dfrac{4\,x^2}{3(a+b)} \div \dfrac{xy}{6(a^2-b^2)}.$

17. $\dfrac{a^2}{a^2-b^2} \div \dfrac{2\,ab}{(a-b)^2}.$

18. $\dfrac{a^3-x^3}{a+x} \div \dfrac{a-x}{-(a+x)^2}.$

19. $\dfrac{x^2-b^2}{-bc} \div \dfrac{b+c}{-b(x+b)^2}.$

20. $\dfrac{2\,ax-x^2}{a(x+a)} \div \dfrac{x^2-a^2}{x+a}.$

21. $\dfrac{-(x+a)^2}{x^2+a^2} \div \dfrac{x-a}{x+a}.$

22. $\dfrac{x^2-1}{x^2-3\,x+2} \div \dfrac{x-1}{x-2}.$

23. $\dfrac{x^2-2\,xy-3\,y^2}{x^2+2\,xy+y^2} \div \dfrac{x-3\,y}{x+y}.$

24. $\dfrac{a^2-ay}{m-n} \div \dfrac{a^4-y^4}{-(a-y)^2}.$

FRACTIONS APPLIED TO EQUATIONS

196. Preparatory.

1. If $\frac{1}{4}$ of a number is 3, what is the number? How is it found? If $(\frac{1}{4})\,x = 3$, what does x equal?

2. If $\frac{3}{4}$ of a number is 6, what is the number? How is it found? If $(\frac{3}{4})\,x = 6$, what does x equal?

3. $(\frac{2}{3})\,y = 10$; what does y equal? $(\frac{3}{2})\,p = 15$; $p = ?$

197. Fractions in Equations. In solving problems by the use of equations the processes with fractions are often used.

EXAMPLE

$\frac{3}{4}$ of the distance from Buffalo to New York City is 330 mi. How far is Buffalo from New York City?

SOLUTION.

Let x be the number of miles in this distance. (*1*)

Then, $\frac{3}{4}x = 330.$ (*2*)

Dividing both members of (*2*) by $\frac{3}{4}$, $x = \frac{4}{3} \cdot 330.$ (*3*)

$\therefore\ x = 440.$

Therefore, the distance from Buffalo to New York is 440 miles. (*4*)

TEST: $\frac{3}{4} \cdot 440 = 330.$

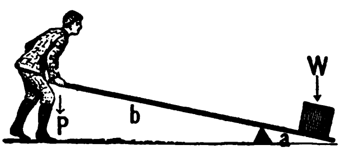
WRITTEN EXERCISES

1. A tennis ball is dropped and rebounds $\frac{2}{3}$ of the distance. From what height is a ball dropped that rebounds $2\frac{2}{3}$ ft.?

2. A church and its tower are together 180 ft. high; the church is one half as high as the tower alone. Find the height of each.

3. The depth of a ship below the water line is $\frac{3}{4}$ as great as the height of the captain's bridge above the water; the bridge is 45 ft. from the bottom of the ship. What is its height above the water?

4. The arms of a lever as shown in the picture are a ft. and b ft. long, and p pounds of force are required to raise a body of weight w. It is



known that the product of one arm of a lever and its corresponding force is equal to the product of the other arm and

its corresponding force. Thus $pb = aw$, from which $p = \frac{aw}{b}$. What force is required to raise 500 lb., if $a = 2$, $b = 10$? Also, if $a = 8$ and $b = 40$? Also, if $a = 9$ and $b = 36$?

5. Find p according to Exercise 4, when $b = a$; how do the force and weight compare when the arms of the lever are equal?

6. Find p when $b = 3a$; p is what part of w in this case?

7. What is the ratio of p to w when the ratio of b to a is 5?

Solve for x and test:

8. $\frac{3}{2}x + 4 = \frac{1}{3} + 10.$

9. $\frac{4}{5}x - 1 = \frac{3}{5} + 8.$

10. $ax + bc = \frac{2b}{c} + 1.$

11. $\frac{a}{b}x = c + d.$

12. $\frac{2}{3}x - \frac{7}{8} = \frac{5}{12}x + \frac{2}{9}.$

13. $\frac{5}{8}x - \frac{2}{3} = x + 4.$

14. $\frac{2}{3}x - 3 = \frac{5}{6}x + 2.$

15. $3x + \frac{7}{8} = \frac{x}{3} - \frac{15}{8}.$

198. In solving equations which contain fractions it is *usually* best to clear of fractions first.

EXAMPLE

Solve:
$$x - \frac{11 - x}{3} = \frac{19 - x}{2}. \qquad (1)$$

Multiplying both members by the l. c. d., 6,
$$6x - \frac{6(11 - x)}{3} = \frac{6(19 - x)}{2}. \qquad (2)$$

Simplifying,
$$6x - 2(11 - x) = 3(19 - x), \qquad (3)$$

and
$$6x - 22 + 2x = 57 - 3x.$$

Uniting terms,
$$11x = 79. \qquad (4)$$
$$\therefore x = 7\tfrac{2}{11}.$$

Test:
$$\frac{79}{11} - \frac{11 - \frac{79}{11}}{3} = \frac{19 - \frac{79}{11}}{2}, \text{ because each } = \frac{65}{11}.$$

199. *To clear an equation of fractions multiply both members by the l. c. d. of all of the fractions.*

WRITTEN EXERCISES

Solve and test:

1. $\frac{y}{2}+\frac{y}{4}-\frac{y}{6}=7.$
2. $\frac{y}{12}-\frac{3-y}{8}=\frac{5+y}{4}-2\frac{3}{4}.$
3. $\frac{x}{3}-\frac{x}{4}=\frac{1}{2}+\frac{x}{5}-\frac{x}{6}.$
4. $\frac{x-1}{2}+\frac{x-2}{3}=6+\frac{x-3}{4}.$
5. $\frac{y}{3}-\frac{y}{4}-\frac{1}{2}=\frac{3y}{4}+1.$
6. $\frac{2x}{3}+\frac{5x}{9}=\frac{x}{6}+\frac{x}{2}+10.$
7. $\frac{x}{5}-\frac{x-8}{4}=\frac{x}{20}.$
8. $\frac{7}{10}x+\frac{2}{5}(20-x)=10.$

200. A fraction that is not given in its lowest terms, should first be reduced.

EXAMPLE

Solve:
$$\frac{x(1+6x)}{x^2-2x}+\frac{1}{x}=6. \qquad (1)$$

Simplifying the first fraction,
$$\frac{1+6x}{x-2}+\frac{1}{x}=6. \qquad (2)$$

Multiplying both members of (2) by the l.c.d.,
$$\frac{x(x-2)(1+6x)}{x-2}+\frac{x(x-2)}{x}=6x(x-2). \qquad (3)$$

Canceling common factors in (3) from numerators and denominators,
$$x(1+6x)+x-2=6x(x-2). \qquad (4)$$

Removing parentheses,
$$x+6x^2+x-2=6x^2-12x. \qquad (5)$$

Simplifying,
$$14x=2. \qquad (6)$$
$$\therefore\ x=\tfrac{1}{7}. \qquad (7)$$

TEST.
$$\frac{\frac{1}{7}(1+\frac{6}{7})}{(\frac{1}{7})^2-2\cdot\frac{1}{7}}+\frac{1}{\frac{1}{7}}=\frac{\frac{1}{7}(\frac{13}{7})}{\frac{1}{7}(\frac{1}{7}-2)}+7, \text{ or } 6=6.$$

WRITTEN EXERCISES

Solve and test:

1. $\frac{1}{3}+\frac{x^2+1}{x-1}=x.$
2. $\frac{x^2-3}{x-1}=x+2.$
3. $\frac{x-2}{x^2-4}=\frac{2}{x-2}.$
4. $\frac{y}{2}-\frac{y^2-1}{y+1}=y.$
5. $\frac{1}{x}-\frac{5}{x^2}=\frac{2x-1}{x^2}.$
6. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{4}{x^2-1}.$
7. $\frac{2}{w-2}-\frac{4}{w+2}=\frac{7}{w^2-4}.$
8. $\frac{3}{x}-\frac{5}{x+6}=\frac{1}{x^2+6x}.$

201. The foregoing processes apply also to fractional equations with two unknowns.

EXAMPLE

Solve for x and y: $\begin{cases} \dfrac{x+1}{y+1}=\dfrac{1}{2}. & (1) \\ \dfrac{x-1}{y-1}=\dfrac{1}{4}. & (2) \end{cases}$

Clearing (1) of fractions, $2x+2=y+1$, or $2x-y=-1$. (3)

Clearing (2) of fractions, $4x-4=y-1$, or $4x-y=3$. (4)

Solving (3) and (4) as in Sec. 166, p. 117, $x=2$ and $y=5$. (5)

WRITTEN EXERCISES

Solve for x and y:

1. $\dfrac{x}{5}+y=3,$
   $x+\dfrac{y}{2}=11.$

2. $\dfrac{x}{3}+\dfrac{3y}{2}=4,$
   $\dfrac{5x}{6}-2y=-13.$

3. $\dfrac{6}{x}+5y=18,$
   $\dfrac{8}{x}-y=1.$

4. $\dfrac{x}{3}+\dfrac{3y}{4}=-4,$
   $\dfrac{x}{y}=\dfrac{3}{4}.$

5. $x+y=5,$
   $\dfrac{x}{y}=\dfrac{1}{4}.$

6. $x+\dfrac{3}{4}-\dfrac{x}{3}-\dfrac{5y}{6}=\dfrac{5x}{9},$
   $\frac{1}{7}(x+y)+\frac{1}{8}(7x-3y)=12.$

7. $\dfrac{x+y}{x-y}=-4,$
   $3x+5y=\dfrac{3}{4}.$

8. $\dfrac{3x+12y}{1+x}=3,$
   $\dfrac{x+4y}{1-4y}=-\dfrac{1}{4}.$

9. $\dfrac{4}{5}\dfrac{(3x-2y)}{(x+2y)}=-\dfrac{8}{10},$
   $\dfrac{1}{3x}+\dfrac{3}{5y}=0.$

10. $\dfrac{20}{x}+\dfrac{4}{3y}=0,$
    $\dfrac{x}{2}-\dfrac{y}{5}=4.$

11. $\dfrac{x+1}{y}=1,$
    $\dfrac{y+1}{x}-1=\dfrac{3}{2}.$

12. $\dfrac{x}{y}=\dfrac{3}{4}$

$x+y+6=\frac{5}{7}.$

13. $\dfrac{11}{6x}=\dfrac{1}{4y},$

$\dfrac{7y}{2x}+4=-13.$

14. Twice the number of pounds of meat consumed annually per inhabitant in Great Britain is 186 % of the consumption per inhabitant in the United States. If the latter is 160 lb., what is the consumption in Great Britain?

15. The number of pounds, as in the previous problem, for the United States falls 39 lb. short of being three times the number for Germany. Find the latter.

16. An average workman should eat daily a certain weight of starchy foods, 16 % of that weight of fats, and 20 % of that weight of albuminous foods (protein). The total weight of these three foods consumed daily should be at least $1\frac{1}{2}$ lb. What weight of each is required?

17. 18 %, by weight, of wheat is lost (as bran, etc.) in grinding into flour. How many 60-pound bushels of wheat are used in making 246 lb. of flour?

18. The weight of bread is $133\frac{1}{3}$ % of the weight of the flour used to make it. According to Exercise 17, how many one-pound loaves of bread can be made from 10 bu. of wheat?

19. The area of Tennessee is $\frac{6}{7}$ of that of New York, and the sum of their areas is 91,000 sq. mi. Find the area of each.

20. The area of Virginia is $\frac{14}{15}$ of that of Pennsylvania, the area of Kentucky is $\frac{8}{9}$ of that of Pennsylvania, and the sum of their areas is 127,000 sq. mi. Find the area of each state.

21. The area of Oklahoma is $\frac{3}{4}$ of that of North Carolina, that of Maryland is $\frac{4}{13}$ of that of Oklahoma, and the sum of their areas is 103,000 sq. mi. Find the area of each.

22. The area of New Hampshire is $\frac{3}{11}$ of that of Maine. 12 times the area of New Hampshire diminished by 3 times the area of Maine is 7000 sq. mi. Find the area of each.

23. A certain number is twice another; their difference divided by their sum equals the smaller. Find the numbers.

24. What fraction becomes equal to $\frac{2}{3}$ if the numerator and the denominator are each increased by 1, and equal to $\frac{1}{2}$ if they are each diminished by 1?

SUGGESTION. Let $\frac{x}{y}$ be the fraction.

25. A fraction whose value is $\frac{5}{6}$ assumes the value $\frac{7}{8}$ if the numerator and the denominator are each increased by 8. Find the fraction.

26. A fraction becomes equal to $\frac{2}{3}$, if the numerator and the denominator are each increased by 3; and equal to $\frac{4}{7}$, if the numerator and the denominator are each diminished by 3. Find the fraction.

27. A certain capital is invested in two kinds of securities, one paying 4 %, the other $4\frac{1}{2}$ %; $\frac{2}{3}$ of the capital is invested in the first kind and the rest in the second; the total income is \$75. What is the capital?

28. A man has an annual income of \$1100 from capital invested partly at 5 % and partly at 6 %. The amount at 6 % was repaid and reinvested at 4 %, and thereafter his annual income was \$800. Find the amounts originally invested.

SUMMARY

I. Definitions and Laws.

1. A fraction answers the questions:

What part is one number of another?
What is the quotient of one number divided by another?
What is the ratio of one number to another? Sec. 170.

2. In algebra, a fraction is usually regarded as an indicated division. Sec. 171.

3. The dividend and the divisor of the indicated division are called the *numerator* and the *denominator* of the fraction. Sec. 172.

4. A fraction is said to be in its *lowest terms* when its numerator and denominator have no common factor. Sec. 173.

5. Every fraction, taken as a whole, has a sign before it, expressed or understood, in addition to the signs that the numerator and the denominator may contain. Sec. 174.

6. When several fractions have the same denominator, that denominator is called their *common denominator*. Sec. 183.

7. Every integer or integral expression may be regarded as a fraction of denominator 1. Sec. 191.

8. If the product of two numbers is 1, each is called the *reciprocal* of the other. Sec. 193.

9. *Law of Signs.* The sign of the fraction is changed if the sign of the numerator or of the denominator is changed. The sign of the fraction is unchanged if the signs of both numerator and denominator are changed. Sec. 175.

10. The *law of exponents* in division applies to fractions. Sec. 180.

11. Multiplying or dividing both numerator and denominator of a fraction by the same number does not change the value of the fraction. Sec. 178.

12. The same factor may be introduced into both numerator and denominator of a fraction without altering its value. Sec. 182.

II. **Processes.**

1. If the numerator of a fraction is of the same degree as the denominator or of a higher degree, the numerator may be divided by the denominator. Sec. 176.

2. A mixed expression may be changed to the fractional form by multiplying the integral part by the denominator of the fractional part and adding the result to the numerator of the fractional part. Sec. 177.

3. Fractions may be reduced to lowest terms by dividing both numerator and denominator by all of their common factors. Sec. 179.

4. To reduce fractions to their *lowest common denominator*, find the l. c. m. of their denominators and multiply the numerator and denominator of each fraction by the quotient of its denominator and the l. c. m. Sec. 184.

5. To add or subtract fractions reduce them to fractions having the l. c. d. and add or subtract the numerators of the resulting fractions. Sec. 186.

6. The product of two or more fractions is the product of their numerators divided by the product of their denominators. Sec. 190.

7. Any power of a fraction equals that power of the numerator divided by the same power of the denominator. Sec. 192.

8. To divide by a fraction is to multiply by its reciprocal. Sec. 194.

9. An equation is cleared of fractions by multiplying both members by the l. c. d. of all the fractions occurring in the equation. Sec. 199.

REVIEW

ORAL EXERCISES

1. A man earned d dollars per week of 6 days. How many dollars did he earn per day? In c days? In d days?

2. An automobile traveled m miles in 5 hr. How far did it travel in 1 hr.? How far would it go in r hours at the same rate? In d days of r hours each?

3. If T tons of coal are delivered in one day by 5 teams, what is the average number of tons per team? How many tons would 7 teams deliver in the same time? t teams?

4. If 200 tons of coal are delivered in 1 day by n teams, how much is this per team? How many tons would 15 teams deliver in the same time? q teams?

WRITTEN EXERCISES

1. Find the value of $\frac{f^2}{s^2}$ when $f=25$, $s=15$.

2. Find the value of $\frac{f}{f-s}$ when $f=20.5$, $s=15.5$.

3. Find the value of $\frac{F-S}{f-s}$ when $F=30$, $S=10$, $f=10$, $s=5$.

4. Find the value of $\frac{mv^2}{R}$ when $m=10{,}000$, $v=90$, $R=300$.

5. A cyclopedia consists of y volumes of p pages each and treats of a subjects. How many subjects is this per page?

6. One capitalist owns $\frac{1}{a}$ of a factory, another owns $\frac{1}{b}$ of it, and another owns $\frac{1}{c}$ of it. What part of it do the three men together own?

7. If the present value of the factory of Exercise 6 is d dollars and it changes to $\frac{a}{b}$ of what it is now, what will be the value of the part owned by these three men? Of the part owned by each man?

8. Answer the questions of Exercise 7, when $a=2$, $b=3$, $c=6$, and $d=\$18{,}000$.

Reduce to an integral or mixed expression:

9. $\frac{x^2+b}{x^2+c}$.
10. $\frac{6x^2+5}{x+5}$.
11. $\frac{x^3-a^3}{x-a}$.

Reduce to fractional form:

12. $a+\frac{b}{c}$.
13. $x+1-\frac{1}{x+1}$.
14. $c+\frac{b^2}{a+b}$.

Add:

15. $\frac{a}{x}-\frac{1}{2x}$.
16. $\frac{x}{12a}-\frac{y}{4}$.
17. $\frac{a-2b}{c}+\frac{3b}{2c}$.
18. $\frac{x-2y}{8y}-\frac{2x-y}{12y}$.
19. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.
20. $\frac{1}{1+x}+\frac{1}{1-x}-\frac{2}{1-x^2}$.
21. $\frac{1}{x+a}+\frac{2}{x+b}-\frac{3}{x+c}$.

22. $\dfrac{5x+4}{x-2}-\dfrac{3x-2}{x-3}-\dfrac{x^2-2x-17}{x^2-5x+6}.$

23. $\dfrac{a-b}{a^2-ab+b^2}+\dfrac{1}{a+b}+\dfrac{ab}{a^3+b^3}.$

24. $\dfrac{(x+y)(x^2+y^2-1)}{xy}+\dfrac{(y+1)(y^2-x^2+1)}{y}$
$+\dfrac{(x+1)(x^2-y^2+1)}{x}.$

Multiply:

25. $\dfrac{x-1}{x}\cdot\dfrac{x+1}{-y}.$

26. $\dfrac{5ax}{3cy}\cdot\dfrac{xy-y^2}{x^2-xy}.$

27. $\dfrac{ax}{(a-x)^2}\cdot\dfrac{a^2-x^2}{-ab}.$

28. $\dfrac{x^2-2xy+y^2}{x+y}\cdot\dfrac{1}{x^2-y^2}.$

29. $\dfrac{x+2}{x-1}\cdot\dfrac{x^2-1}{x^2-4}.$

30. $\dfrac{a+b}{-b}\cdot\dfrac{ac}{a^2-b^2}.$

31. $\dfrac{x^2-9}{x^2+4x}\cdot\dfrac{x^2-16}{x^2-3x}.$

32. $\dfrac{4x^2}{3(a+b)}\cdot\dfrac{6(a^2-b^2)}{-xy}.$

33. $\dfrac{-4ab}{6c}\cdot\dfrac{9c^2d}{-7a^2b}\cdot\dfrac{-21c}{12abc}.$

34. $\dfrac{a-b}{a^2+2ab}\cdot\dfrac{a^2-4b^2}{a^2-ab}.$

Divide:

35. $\dfrac{a^2}{b}\div\dfrac{a}{b^2}.$

36. $\dfrac{ac}{bd}\div\dfrac{ad}{bc}\div\dfrac{x^2}{y}\div\dfrac{x}{y^2}.$

37. $\dfrac{a+b}{a-b}\div\dfrac{(a+b)^2}{a-b}.$

38. $\dfrac{x^2+5x+6}{x^2-1}\div\dfrac{x+5}{x+1}.$

39. $\left(\dfrac{ax}{by}\right)^2\div\left(\dfrac{cy}{bx}\right)^3.$

40. $\left(\dfrac{a-b}{x-y}\right)^3\div\left(\dfrac{a^2-b^2}{x^2-y^2}\right)^2.$

41. $\left(\dfrac{m-n}{m-p}\right)^4\div\left(\dfrac{n-m}{n-p}\right)^4.$

42. $\dfrac{x}{1+x}\div\dfrac{x+x^2}{(1+x)^2}.$

43. The number 128 is the sum of two numbers such that $\frac{1}{7}$ of one equals $\frac{1}{25}$ of the other. What are these numbers?

44. A certain kind of woolen cloth 1 yd. wide shrinks $\frac{1}{20}$ of its length and $\frac{1}{16}$ of its width in washing. How many yards must be bought in order to have 38 sq. yd. of cloth after shrinking?

SUPPLEMENTARY WORK

Addition of Fractions

Fractions in which the letters are symmetrically involved are often easily added.

EXAMPLE

Add $\dfrac{ab}{(c-a)(c-b)}+\dfrac{bc}{(a-b)(a-c)}+\dfrac{ca}{(b-c)(b-a)}$. (*1*)

The l. c. d. is $(a-b)(b-c)(c-a)$. Sec. 184, p. 131.

Since $-(a-b)=b-a$, and $-(c-b)=b-c$,

$$\frac{ab}{(c-a)(c-b)}=\frac{ab(b-a)}{(c-a)(b-c)(a-b)}. \quad (2)$$

$$\frac{bc}{(a-b)(a-c)}=\frac{bc(c-b)}{(a-b)(b-c)(c-a)}. \quad (3)$$

$$\frac{ca}{(b-a)(b-c)}=\frac{ca(a-c)}{(a-b)(b-c)(c-a)}. \quad (4)$$

$$\therefore \frac{ab}{(c-a)(c-b)}+\frac{bc}{(a-b)(a-c)}+\frac{ca}{(b-c)(b-a)} \quad (5)$$

$$=\frac{ab(b-a)+bc(c-b)+ca(a-c)}{(a-b)(b-c)(c-a)}=1.$$

The numerator and the denominator are identical when expanded. It may be noticed that the second fractions in steps (*2*), (*3*), and (*4*) are of the same form and that all may be inferred when the first has been found.

Test by substituting $a=1,\ b=2,\ c=3$.

WRITTEN EXERCISES

Add:

1. $\dfrac{a}{c(a-b)}-\dfrac{c}{a(b-a)}$. 2. $\dfrac{1}{(a-b)(c-a)}-\dfrac{1}{(b-a)(c-b)}$.

3. $\dfrac{a}{(b-c)(c-a)}+\dfrac{b}{(c-a)(a-b)}+\dfrac{c}{(a-b)(b-c)}$.

4. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}$.

5. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}$.

6. $\dfrac{a(b+c)}{(a-b)(c-a)}+\dfrac{b(c+a)}{(b-c)(a-b)}+\dfrac{c(a+b)}{(c-a)(b-a)}$.

7. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$.

8. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-z)(y-x)}+\dfrac{1}{z(z-x)(z-y)}$.

Fractions may sometimes be added more easily by adding them in an order different from that in which they were given.

EXAMPLES

1. In adding $\dfrac{1}{a-1}+\dfrac{1}{a-3}+\dfrac{-1}{a+1}+\dfrac{1}{a+3}$ it is especially easy to add the first and third fractions; then the second and fourth; and, finally, the results thus obtained.

Thus,

$$\frac{1}{a-1}-\frac{1}{a+1}=\frac{a+1-a+1}{a^2-1}=\frac{2}{a^2-1}.$$

$$\frac{1}{a-3}+\frac{1}{a+3}=\frac{a+3+a-3}{a^2-9}=\frac{2a}{a^2-9}.$$

$$\frac{2}{a^2-1}+\frac{2a}{a^2-9}=\frac{2a^2-18+2a^3-2a}{(a^2-1)(a^2-9)}=\frac{2(a^3+a^2-a-9)}{(a^2-1)(a^2-9)}.$$

2. In adding $\dfrac{3}{a-b}+\dfrac{3}{a+b}+\dfrac{ba}{a^2+b^2}+\dfrac{12a^3}{a^4+b^4}$ the first and second are easily combined, then that result is easily added to the third, and finally that result to the fourth.

Thus:

$$\frac{3}{a-b}+\frac{3}{a+b}=\frac{6a}{a^2-b^2}.$$

$$\frac{6a}{a^2-b^2}+\frac{6a}{a^2+b^2}=\frac{12a^3}{a^4-b}.$$

$$\frac{12a^3}{a^4-b^4}+\frac{12a^3}{a^4+b^4}=\frac{24a^7}{a^8-b^8}.$$

WRITTEN EXERCISES

1. Add the fractions in the examples above in the ordinary way; that is, find the l. c. d. of the several fractions and add at once.

Add by successive combinations, as above:

2. $\dfrac{1}{a-x}+\dfrac{1}{a+x}-\dfrac{2a}{a^2+x^2}.$

3. $\dfrac{1}{a-1}+\dfrac{3}{a-2}-\dfrac{3}{a+2}-\dfrac{1}{a+1}.$

4. $\dfrac{2}{a-b}+\dfrac{2}{a+b}+\dfrac{4a}{a^2+b^2}.$

5. $\dfrac{1}{a-b}+\dfrac{a-b}{c}-\dfrac{1}{a+b}-\dfrac{a+b}{c}.$

6. $\dfrac{1}{p+1}-\dfrac{2}{p-2}+\dfrac{2}{p+2}-\dfrac{1}{p+1}.$

7. Add several of the preceding problems in the ordinary way. Which method is shorter?

Complex and Simple Fractions

Since a fraction indicates division, the division of two fractions may be indicated in fractional form.

Thus, $\frac{3}{8}\div\frac{7}{8}$ may be written $\dfrac{\frac{3}{8}}{\frac{7}{8}}$,

and $\frac{a}{b}\div\frac{c}{d}$ may be written $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.

Complex Fractions. If either the numerator or the denominator of a fraction is fractional in form, or if both are so, the fraction is called a **complex fraction**.

For example: $\dfrac{\frac{a+b}{c}}{d},\ \dfrac{2}{\frac{7}{3}},\ \dfrac{\frac{31}{2+a}}{\frac{x+2}{3}},\ \dfrac{\frac{3x}{4y}}{\frac{7x}{12y}}.$

Simple Fractions. In distinction from complex fractions, **simple fractions** are those in which neither the numerator nor the denominator is in fractional form.

A complex fraction may be reduced to a simple fraction either by performing the indicated divisions, or by multiplying both numerator and denominator by the l. c. m. of their respective denominators. In each case any operations indicated in the numerator or the denominator should be performed first, as far as possible.

EXAMPLES

1. Simplify: $\dfrac{\dfrac{x^2-y^2}{a+b}}{\dfrac{x+y}{a^2-b^2}}$.

$$\frac{\dfrac{x^2-y^2}{a+b}}{\dfrac{x+y}{a^2-b^2}}=\frac{x^2-y^2}{a+b}\div\frac{x+y}{a^2-b^2}=\frac{x^2-y^2}{a+b}\times\frac{a^2-b^2}{x+y}=(x-y)(a-b).$$

2. Simplify: $\dfrac{\dfrac{a}{ax}+\dfrac{b}{ax}}{\dfrac{b+c}{by}}$.

$$\frac{\dfrac{a}{ax}+\dfrac{b}{ax}}{\dfrac{b+c}{by}}=\frac{\dfrac{a+b}{ax}}{\dfrac{b+c}{ay}}=\frac{\dfrac{axy(a+b)}{ax}}{\dfrac{axy(b+c)}{ay}}=\frac{y(a+b)}{x(b+c)}.$$

WRITTEN EXERCISES

Reduce to simple fractions:

1. $\dfrac{\dfrac{24x}{y}}{6x}$.

2. $\dfrac{4}{\dfrac{8}{a^2}}$.

3. $\dfrac{\dfrac{7a}{5c}}{21b}$.

4. $\dfrac{m^2-9}{\dfrac{m+3}{a}}$.

5. $\dfrac{\dfrac{1}{a-b}}{a+b}$.

6. $\dfrac{\dfrac{12ab}{35cd}}{\dfrac{25d}{16ac}}$.

7. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{x-y}$.

8. $\dfrac{1-49t^2}{\dfrac{7t+1}{3ab}}$.

9. $\dfrac{\dfrac{x^2-1}{5a}}{\dfrac{x-1}{15a^2b}}$.

10. $\dfrac{2+ax+\dfrac{1}{ax}}{a^2x^2-1}$.

11. $\dfrac{\dfrac{1}{x-y}-\dfrac{1}{x+y}}{\dfrac{y}{x-y}}$.

12. $\dfrac{\dfrac{2ax}{a+x}-a}{\dfrac{1}{x}+\dfrac{1}{a-2x}}$.

13. $\dfrac{\dfrac{a-3}{3b}-\dfrac{3b}{a-3}}{\dfrac{1}{3b}-\dfrac{1}{a-3}}$.

14. $\dfrac{\dfrac{a}{b^2}+\dfrac{b}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ab}+\dfrac{1}{b^2}}$.

15. $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}$.

16. $\dfrac{\dfrac{a^2-3ab}{x^2-1}}{\dfrac{9b^2-6ab+a^2}{1+x}}$.

17. $\dfrac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b}{a}-1}-\dfrac{1+\dfrac{b^2}{a^2}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b^2}{a^2}}$.

18. $\dfrac{\dfrac{ab}{c}+\dfrac{bc}{a}+\dfrac{ca}{b}}{\dfrac{a}{bc}+\dfrac{b}{ca}+\dfrac{c}{ab}}\times\left[\dfrac{(a+b+c)^2}{ab+bc+ca}-2\right]$.

19. $\dfrac{x^4+x^3-x-1}{1-y^2}\cdot\dfrac{y^2-1}{x^2-x}\cdot\left(1-\dfrac{1}{1-\dfrac{1}{x}}\right)$.

Simplify and solve for x:

20. $\dfrac{\dfrac{13x+2}{4}}{\dfrac{5x+11}{2}}=1.$ 21. $\dfrac{\dfrac{5x}{3}}{\dfrac{4x+1}{2}}=-3.$ 22. $\dfrac{\dfrac{7+x}{3+x}}{\dfrac{x+1}{x-1}}=1.$

23. $\dfrac{5+\dfrac{12}{x}}{7-\dfrac{4}{x}}=\dfrac{\dfrac{48}{x+2}}{\dfrac{36}{x+2}}$.

24. $\dfrac{\dfrac{7-3x}{5+9x}}{\dfrac{8}{5+9x}}=\dfrac{\dfrac{7-3x}{48x}}{4+6x}$.

CHAPTER XII

RATIO, PROPORTION, AND VARIATION

RATIO

202. Ratio. The quotient of two numbers of the same kind is often called their **ratio.**

The following examples illustrate the use of the word "ratio":

The ratio of 12 qt. to 4 qt. is 3.

A solution consists of sulphuric acid and water in the ratio of 2 to 3. This means that $\frac{2}{5}$ of the whole is sulphuric acid and $\frac{3}{5}$ is water.

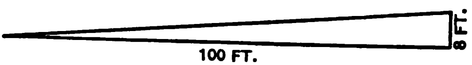
Sterling silver requires a little other metal (alloy) to harden the silver in it; the ratio by weight of the amount of pure silver to the entire mass is usually 925 to 1000. This means that $\frac{925}{1000}$ of the whole is silver.

If a body moves uniformly, the rate of motion in feet per second is the ratio of the number of feet moved to the number of seconds required.

The specific gravity of a solid is the ratio of its weight to the weight of an equal volume of water.

The rate of interest on an investment is the ratio of the number of dollars of interest received to the number of dollars invested.

The birth rate per annum in a city is said to be 23 when the ratio of the total number of births in a certain year to the total number of inhabitants at the beginning of that year is that of 23 to 1000.



A road bed is said to have an 8% grade when the ratio of the vertical rise to the horizontal distance is that of 8 to 100.

203. Ratio is commonly expressed by a fraction.

Thus, the ratio of 2 to 3 is written $\frac{2}{3}$; the old form 2 : 3 is less convenient in calculation.

In division the divisor may be abstract or concrete. If it is abstract, the quotient is of the same character as the dividend. If it is concrete, the dividend must be concrete and expressed in the same unit, and the quotient is abstract.

Consequently, the ratio of 12 qt. to 3 qt. is 4, the ratio of 12 to 3 is 4. We cannot speak directly of the ratio of 12 gal. to 3 qt.; we must first express both numbers in the same unit, as 48 qt. to 3 qt.

Similarly, when we speak of the ratio of the distance to the time, we mean the ratio of the corresponding abstract numbers, as in Sec. 202.

WRITTEN EXERCISES

1. How much pure silver in 200 oz. of sterling silver (Sec. 202)?

SUGGESTION. $\frac{x}{200} = \frac{925}{1000}$.

2. A silversmith buys $46\frac{1}{4}$ oz. of pure silver; how much sterling silver (Sec. 202) can be made from it?

SUGGESTION. The equation is $\frac{46\frac{1}{4}}{x} = \frac{925}{1000}$.

3. What is the rate per second of a train which travels uniformly 635 ft. in 5 sec.?

4. The weight of a piece of gold is 94.5 oz., and the weight of an equal volume of water is 5 oz. What is the specific gravity (Sec. 202) of the gold?

5. There are 4053 births in a certain city, making its birth rate 21 (Sec. 202). What is the population of the city?

6. If the population of a city is p and the birth rate is b, indicate the number of births.

7. The top of a mountain pass is 1200 feet vertically above the level of the base; the top is reached by a zigzag road 5 mi. long. What is the average grade of the road?

8. A road m miles long ascends to a height of f feet above the level of its starting point. Indicate the average grade of the road.

9. Two men, A and B, divide \$963 of profits so that A's part is to B's in the ratio of 2 to 1. How many dollars has each?

SOLUTION.
1. Let x be the amount A receives.
2. Then, $963 - x$ is the amount B receives.
3. $\therefore$ $\frac{x}{963 - x} = \frac{2}{1}$, the ratio of the shares, as given.
4. Clearing of fractions, $x = 2(963 - x)$, and $x = 642$.
5. $\therefore$ A receives \$642 and B receives \$321.

TEST. \$642 + \$321 = \$963, and $\frac{642}{321} = \frac{2}{1}$.

10. Two partners, A and B, divide \$575 in the ratio of 2 to 3. How many dollars does each receive?

11. Supply the blanks in the table:

	Substance	w = weight of substance	w' = weight of an equal volume of water	Specific gravity $= \frac{w}{w'}$
(1)	Lead	56.5 oz.	5 oz.	——
(2)	Oak	12.75 oz.	15 oz.	——
(3)	Tin	153.3 lb.	21 lb.	——
(4)	Coal	18 tons	10 tons	——
(5)	Alcohol	13.6 oz.	17 oz.	——

12. It is known that in triangles whose corresponding angles are equal, the corresponding sides have the same ratio.

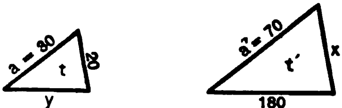


FIG. 1.

In Fig. 1, what is the ratio of a to a'? Of 20 to x? Find x. What is the ratio of y to 180? Find y.

13. What is the ratio $\frac{h}{l}$ as shown in the diagram of an approach to a bridge? $\frac{h'}{80}$ has the same value. Find h'.

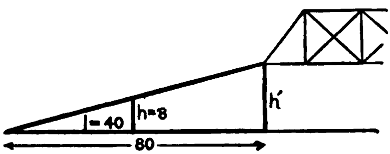


FIG. 2.

PROPORTION

204. Proportion. An equation between two ratios is called a **proportion.**

Thus, $\frac{2}{3}=\frac{6}{9}$, $\frac{3a}{7a}=\frac{6b}{14b}$, $\frac{25}{5}=\frac{5}{1}$, $\frac{5}{6}=\frac{3}{x}$ are proportions.

A proportion is usually read in one of two ways: For example, $\frac{3}{5}=\frac{a}{x}$ is read "3 is to 5 as a is to x," or "the ratio three-fifths equals a over x."

205. The numbers forming one of the ratios are said to be "proportional to" the numbers forming the other.

Thus, 12 and 9 are proportional to 20 and 15, because $\frac{12}{9} = \frac{20}{15}$.

206. The terms "proportional" and "proportionally" are used with the meaning "in the same ratio."

For example:

The express rate from Chicago to New York is $2.50 per 100 lb., the excess above 100 lb. being charged proportionally. This means the charge for the excess has the same ratio to the excess as 2.50 has to 100.

Of two men in business one furnished $\frac{2}{3}$ of the capital, and the other $\frac{1}{3}$ of it. They divided their gain of $9000 in proportion to their capitals. This means that they divided the $9000 into two parts having the ratio of 2 to 1.

ORAL EXERCISES

Explain these statements according to Sec. 206:

1. In Mr. Brown's store sugar costs $\frac{1}{5}$ more than in Mr. Wilson's and other things in proportion.

2. A train travels so that the distance traveled is proportional to the time.

3. Two families of 3 members and 5 members respectively camp out together at an expense of $160; they divide this amount in proportion to the size of the families.

207. The expression *pro rata* is often used with the same meaning as "proportionally" or "in the same ratio."

For example, A and B hire an automobile for a trip and agree to pay $\frac{3}{5}$ and $\frac{2}{5}$ of the rental respectively, and other expenses occurring on the trip *pro rata*. This means that the other expenses are to be divided between A and B in the same ratio as the rental of the automobile.

WRITTEN EXERCISES

1. In the example in Sec. 207, the other expenses are $40. How many dollars of this must A pay?

2. A man hires a piano at $5 per month of 30 days; and is to pay pro rata for any part of a month. What does he pay, if he keeps the piano 51 days? *d* days?

3. Three farmers share in the purchase of a steam thresher. A pays \$400, B pays \$600, and C pays \$1000. In the course of the year the thresher is rented to other farmers 27 days at \$10 per day, and the earnings divided among the owners pro rata. What does each receive?

4. If the thresher of Exercise 3 is rented d days at r dollars per day and the earnings divided pro rata, what does each owner receive?

5. It costs R dollars to repair the thresher, and the cost is divided pro rata among the owners. What does each pay?

6. If the rental in Exercise 4 just defrayed the repair in Exercise 5, express d in terms of R and r. If the repairs cost \$150, and the daily rental was \$12, for how many days was it rented?

Note. It will be noticed that there is no new mathematical process in Ratio and Proportion. But the pupil must be familiar with these terms, because they are names used for quotients and the equality of quotients not only in mathematics, but also in other sciences and in business.

ORAL EXERCISES

Find the value of x in the following proportions:

1. $\frac{2}{3}=\frac{4}{x}$. 2. $\frac{3}{5}=\frac{x}{15}$. 3. $\frac{x}{b}=\frac{3a}{3b}$. 4. $\frac{x}{3}=\frac{4}{6}$.

WRITTEN EXERCISES

1. Find x in the proportion $\frac{x}{15}=\frac{160}{25}$. (1)

Solution:

Multiplying both members by 15, $x=\frac{15 \times 160}{25}$. (2)

Simplifying the fraction in (2), $x = 96$. (3)

2. Find x in the proportion $\frac{7}{x}=\frac{.14}{42}$. (1)

Solution:

Clearing of fractions, $7 \times 42 = .14\,x$. (2)

Simplifying (2), $x=\frac{7 \times 42}{.14}=2100$. (3)

3. Solve the proportion $\frac{h}{l}=\frac{h'}{l'}$ for h. For h'. For l'.

4. Solve the proportion $\frac{p}{P}=\frac{W}{w}$ for p. For P. For W. For w.

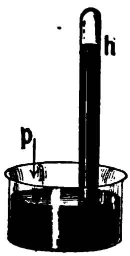
5. Solve the equation $\frac{p}{p'}=\frac{b}{b'}$ for p. For b. For b'. For p'.

6. In Exercise 5 what is the value of p', if $p = 15$, $b = 28$, and $b' = 30.5$?

7. Two partners, A and B, in business divide $9000 between them in the ratio of 2 to 1. How much does each receive?

8. At $2.80 for 8 hours' work, overtime paid proportionally, how much does a workman receive for $2\frac{1}{2}$ hr. overtime?

9. After rents rose $\frac{1}{8}$ and other things in proportion, a family's expenses for one month were $132. What were their expenses before the rise?



10. The height to which the mercury rises in a barometer is proportional to the pressure of the air on the mercury. Let the heights at two readings be h and h', and the corresponding pressure p and p'; then $\frac{p}{p'}=\frac{h}{h'}$. Express p in terms of h, h', and p'.

11. In Exercise 10, let two readings of the barometer be $h = 28.5$ and $h' = 29.5$, and the corresponding pressures be p and $p' = 15$ lb. What is the value of p?

12. Supply the blanks in the following table of barometric readings:

	p	p'	h	h'
(1)	( ) lb.	14.5 lb.	30 in.	29 in.
(2)	14.75 lb.	( )	29.5 in.	30 in.
(3)	15 lb.	15.25 lb.	( )	30.5 in.
(4)	14.2 lb.	14.8 lb.	28.75 in.	( )

13. The point of support of a lever is called the fulcrum and in the figure is denoted by F.

If P denotes the power and W the weight (expressed in the same unit), and if p denotes the length of the arm FP and w the length of the arm FW (both expressed in the same unit), then it is known that

$$\frac{w}{p}=\frac{P}{W}.$$

In words: *The ratio of the lengths of the arms equals the reciprocal of the ratio of the corresponding forces.*

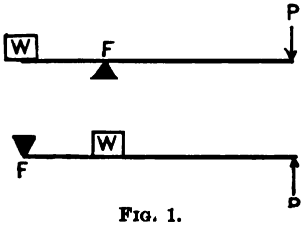


FIG. 1.

Express w in terms of P, W, and p. Express W in terms of P, p, and w.

14. Find p if $P=10$ lb., $W=60$ lb., $w=2$ ft.

15. Supply the numbers to fill the blanks in the following table concerning levers:

	p	w	P	W
(1)	( )	20 in.	1.60 lb.	90 lb.
(2)	1.5 ft.	( )	2.25 lb.	1125 lb.
(3)	12 ft.	$\frac{1}{2}$ ft.	( )	960 lb.
(4)	.3 ft.	12.9 ft.	2.5 T.	( )

16. Areas of similar triangles (those of the same shape) are proportional to the squares of the lengths of their corresponding sides. Let A and A' be the areas of two similar triangles, and a and a' be a pair of corresponding sides. Then $\dfrac{A}{A'}=\dfrac{a^2}{a'^2}$.

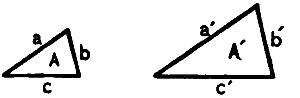


FIG. 2.

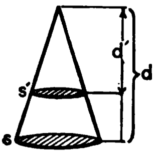
Solve the proportion $\dfrac{A}{A'}=\dfrac{a^2}{a'^2}$ for A. For A'. For a. For a'.

17. Express the ratio of a to a' in Exercise 16.

18. The areas of two similar triangles are 64 sq. in. and 625 sq. in. What is the ratio of any pair of corresponding sides?

19. If the side a of the smaller triangle in Exercise 16 is 8 in., what is the corresponding side a', in the larger triangle?

20. The lengths of a pair of corresponding sides in two similar triangles are 3 ft. and 8 ft.; the area of the larger one is 640 sq. ft. What is the area of the smaller one?



21. It is known that in any cone the areas of parallel sections are proportional to the squares of their distances from the vertex. Thus, in the figure, $\frac{s'}{s} = \frac{d'^2}{d^2}$.

If $d = 1$, $d' = \frac{1}{2}$, and $s = 40$ sq. in., what is the area of s'?

22. The area of a section $\frac{1}{3}$ of the way from the vertex to the base and parallel to it is what part of the base?

23. 1 cu. ft. of lime and 2 cu. ft. of sand are used in making 2.4 cu. ft. of mortar. How much of each is needed to make 72 cu. ft. of mortar?

24. Mortar for use under water may be made of 1 part of lime, 1 part of cement, and 8 parts of sand (by bulk). How much of each is required to make 90 cu. ft. of mortar, assuming that, in mixing, the volume of the materials is reduced by $\frac{1}{3}$?

25. In making glass, 15, 5, and 1 portions by weight of sand, potash, and chalk respectively are used. How many pounds of each are required to produce 1176 lb. of glass?

VARIATION

208. If related numbers change so as always to remain in the same ratio, one is said to *vary as*, or *vary directly with*, the other.

For example:

At 5¢ per pound, the amount paid varies as the number of pounds purchased.

If a body moves at the same rate, the distance varies as the time of motion.

If $100 is placed at simple interest, the amount of interest varies as the time.

209. "Varies as" is thus seen to be merely another expression for "is proportional to" or "varies proportionally with."

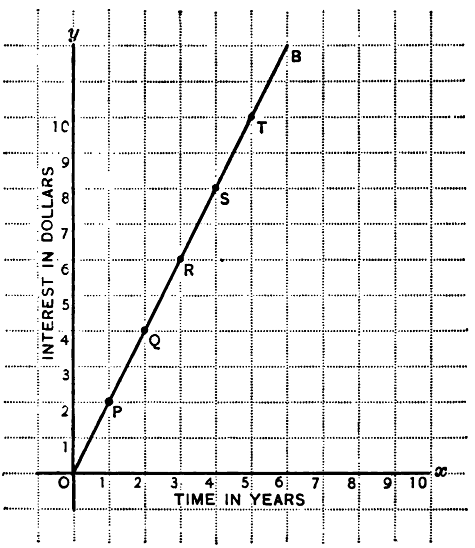
210. If y varies as x, and c denotes the fixed ratio, then, whatever value x has, the corresponding value of y must be so related to it that $\frac{y}{x} = c$, or $y = cx$.

211. Formula for Direct Variation. $y = cx$ is the formula that expresses the relation of **direct variation** between y and x.

212. Direct variation can be represented graphically.

EXAMPLES

1. If \$100 is placed at 2 % simple interest, show graphically how the amount of interest (I) varies with the number of years (t).



In the diagram the spaces on the horizontal scale represent the number of years, and those on the vertical scale the number of dollars. The positions of the points on the line OB show the amounts of interest for the periods of time indicated on the horizontal scale. Thus, point P shows that the interest is \$2 when the time is 1 yr.; point Q shows that the interest is \$4 when the time is 2 yr.

2. What does point R show? Point O? Point S? Point T?

3. What does the point halfway between Q and R show?

The number of dollars interest is always twice the number of years. This is expressed by the formula, $I = 2t$.

Every point in the line OB shows the interest on \$100 at 2 % for the number of years indicated by its distance to the right of the line OY.

From the graph we can read either the amount of interest for a given time, or the time required to earn a given interest.

ORAL EXERCISES

From the graph read:

1. The amount of interest on \$100 at 2 % for 2 yr. For 3 yr. For $1\frac{1}{2}$ yr.

2. The time in which \$100 at 2 % will earn \$4. \$5. \$3.

WRITTEN EXERCISES

1. Make a graph like that on page 168, taking the rate of interest to be 3 %.

2. From the graph answer questions like 1 and 2 above. State the relation between interest and time in this case. Write an equation expressing this relation.

3. Treat similarly each of the following rates: 4 %; $2\frac{1}{2}$ %; 5 %; 6 %.

4. Using horizontal spaces to represent numbers of pounds, and vertical spaces to represent 10 ¢ each, mark points to indicate the cost of 3 lb. of soda at 10 ¢ a lb. The cost of 5 lb. Of 8 lb. Of 10 lb. Draw a line through the points. It will represent the cost of various numbers of pounds at 10 ¢ per pound. From the graph read the cost of 4 lb.; 6 lb.; 9 lb.

5. Using horizontal spaces to represent numbers of miles, and vertical spaces to represent cents, mark points to represent the cost of railway tickets at 3 ¢ per mile for the following numbers of miles: 1 mi.; 2 mi.; 5 mi.; 10 mi. Draw a line through the points, and by use of it read the cost of tickets for the following distances: 4 mi.; 6 mi.; 8 mi.; 9 mi; 5 mi.

6. An elevator goes up at the rate of 4 ft. per sec. What distance does it ascend in 2 sec.? In 3 sec.? In 5 sec.? In 1 min.?

7. Letting $d=$ the number of feet passed over in any number of seconds, and using horizontal spaces to represent the number of seconds, and vertical spaces to represent the number of feet, make a graph to represent the relation $d=4t$.

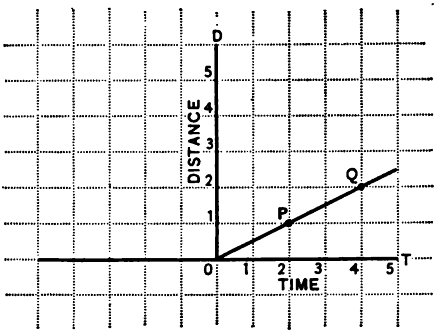


FIG. 1.

8. In Fig. 1 values of t are represented on the horizontal line OT, and corresponding values of d on the line OD. What is the value of t for point P? Of d for point P? What is the value of t for point Q? Of d for point Q? Every value of d is what part of the corresponding value of t? Express this relation by an equation.

213. Before constructing a graph, the corresponding values of the letters may be conveniently arranged in a table as in the following example:

Construct the graph of $y=4x$.

TABLE

x	y
0	0
1	4
$1\frac{1}{2}$	6
2	8
$2\frac{1}{2}$	10
3	12
4	16
5	20

GRAPH

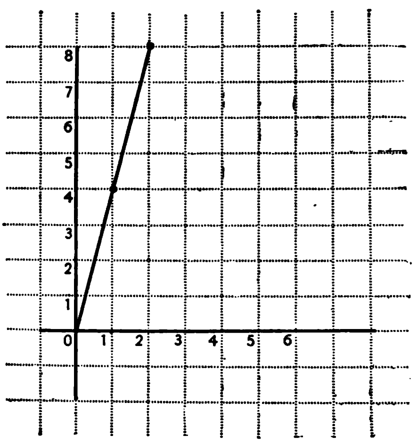


FIG. 2.

WRITTEN EXERCISES

Construct the graphs of:

1. $2y=3x$.	6. $\frac{1}{2}y=x$.	11. $d=5t$.
2. $4y=x$.	7. $y=\frac{3}{2}x$.	12. $\frac{1}{3}d=t$.
3. $y=\frac{1}{3}x$.	8. $y=\frac{2}{3}x$.	13. $d=3\frac{1}{2}t$.
4. $y=\frac{1}{2}x$.	9. $d=\frac{1}{4}t$.	14. $2\frac{1}{2}y=x$.
5. $y=\frac{4}{5}x$.	10. $d=\frac{3}{4}t$.	15. $1\frac{1}{4}x=y$.

214. Graphs may be constructed for negative values as well as for positive values of the numbers involved.

EXAMPLES

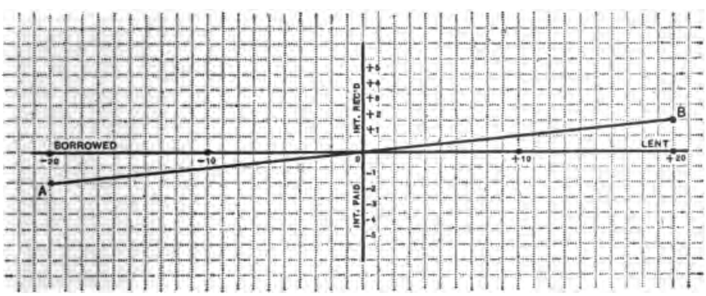
1. If we regard money borrowed (which is the opposite to money lent) as negative, and interest paid (which is the opposite to interest received) as negative, we may express by a single line the changes in interest and principal, both for money borrowed and for money lent.

The table shows the change in interest at 5% for 2 yr. corresponding to the change in the principal from + \$20 to − \$20, the positive values denoting money lent and interest received, the negative ones denoting money borrowed and interest paid. The line AOB in the figure represents these changes.

TABLE

$p=$	20	10	0	−10	−20
$i=$	2	1	0	−1	−2

GRAPH



2. The line CB in Fig. 1 is the graph of the equation $y = 2x$, for both positive and negative values. The line OB is the graph for all positive values of x and y and the line OC, the extension of OB, is the graph for negative values. The negative values of x are marked off to the left of O and the negative values of y downward from O.

TABLE

y	x
-4	-2
-3	$-1\frac{1}{2}$
-2	-1
-1	$-\frac{1}{2}$
0	0
1	$\frac{1}{2}$
2	1
3	$1\frac{1}{2}$
4	2

GRAPH

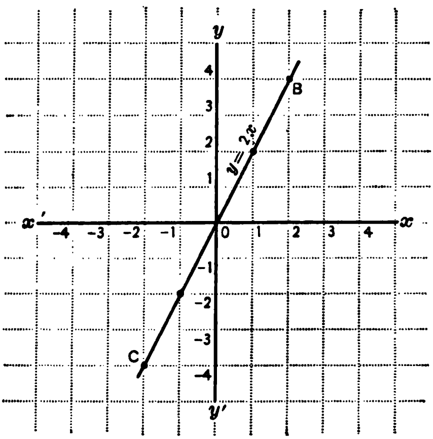


FIG. 1.

215. Any change that takes place throughout at a constant rate is said to take place "uniformly."

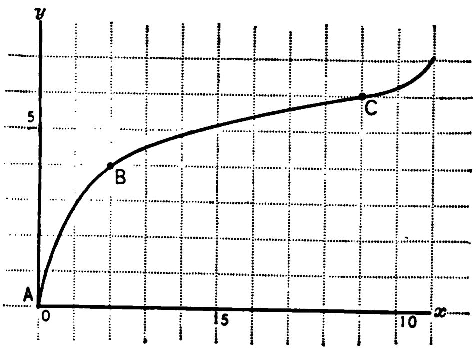


FIG. 2.

For example, a body moving continually at the same speed is said to move uniformly. That *the graph of uniform change* will be a straight line appears from the fact that the straight line is the only path along which a point moves upward or downward uniformly. Along any curved line it moves upward or downward more rapidly at some times than at others. Thus, in the figure the increase from A to B is much more rapid than from B to C.

If x and y vary, subject to the equation $y = 4x + 3$, or to $y = mx + b$, the change will be uniform. For, in the first case, y changes by 4 times the change in x, and, in the second case, by m times the change in x; that is, y varies uniformly with x. Since we know that only straight lines represent uniform change, we know that the graphs of the equations above are straight lines.

ORAL EXERCISES

1. How many points are necessary to fix the position of a straight line?

2. How many points must be fixed in order to draw a graph which is known to be a straight line?

216. Constructing the graph of an equation is called **plotting** the equation.

WRITTEN EXERCISES

Plot the following equations:

1. $3y = x$.	4. $y = \frac{1}{3}x$.	7. $2x - y = 0$.
2. $y = 3x$.	5. $p = 4t$.	8. $5p - w = 0$.
3. $c = \frac{1}{2}p$.	6. $s = \frac{1}{4}t$.	9. $x - 6y = 0$.

SUMMARY

Definitions.

1. A *ratio* is the quotient of two numbers of the same kind. Sec. 202.

2. A *proportion* is an equation between two ratios. Sec. 204.

3. The numbers forming one ratio of a proportion are said to be *proportional to* the numbers forming the other ratio. Sec. 205.

4. "Proportionally" and "pro rata" mean "in the same ratio." Sec. 207.

5. If two related numbers vary so as always to remain in the same ratio, one is said to *vary as* the other. Other expressions for "varies as" are "varies directly with" and "is proportional to." Sec. 208.

6. $y = cx$ is the formula for direct variation. Sec. 211.

7. *Plotting an equation* is constructing its graph. Sec. 216.

REVIEW

WRITTEN EXERCISES

1. When the grade of a roadbed is $8\frac{1}{2}\%$, what is the rise in a horizontal distance of 175 ft.?

2. The specific gravity of cast iron is 7.21. If a cubic foot of water weighs $62\frac{1}{2}$ lb., find the weight of a cubic foot of iron.

3. At $p\%$ interest the annual income from a certain investment is D dollars. Indicate the amount of the investment.

4. What weight can a man weighing 175 lb. raise with a levor 5 ft. long, if the weight is applied 6 in. from the fulcrum?

5. What weight can he raise, if he weighs 180 lb., the lever being $4a$ ft. long, and the support being a in. from the end acting on the weight?

Plot:

6. $y = 3x$.

7. $5x - y = 0$.

8. $2x = y$.

9. $x - 2y = 0$.

10. $d = 5t$.

11. $s = \frac{1}{2}t$.

CHAPTER XIII

FACTORING

217. Various methods of finding the factors of algebraic expressions have already been treated in the chapter on multiplication. The following is a summary and extension of the methods given before.

I. MONOMIAL FACTORS

218. When every term of a polynomial contains a common factor, that factor may usually be found by inspection.

Sec. 117, p. 77.

For example:

$3ab$ is a factor of $3abx - 6aby - 9abz$, for $3abx - 6aby - 9abz = 3ab \cdot x - 3ab \cdot 2y - 3ab \cdot 3z$.

ab is a factor of $a^2b + ab^2 - abc$, for $a^2b + ab^2 - abc = ab \cdot a + ab \cdot b - ab \cdot c$.

ORAL EXERCISES

State the monomial factor of each expression:

1. $ab + ac + ad$.
2. $ab + bc + b$.
3. $2ax + 2ay + 2az$.
4. $m^2 + m^2y + m^2z$.
5. $3mx - 6my - 9mz$.
6. $5ab + 10a^2b^2 - 5abc$.
7. $2a(x - y) + 2axy$.
8. $a^2b^2 - 3ab^3 + 5a^3b$.
9. $10^{n+3} + 10^{n+5} + 10^n$.
10. $\frac{2ab}{3} - \frac{a^2}{6} + \frac{5ac}{12}$.
11. $6a^2b^2c^2 - 2a^3bc - 2abc$.
12. $3ay^2 + 6a^2y - 9a^3y^3$.
13. $x^3 - 6x^2 + 12x$.
14. $x^2y^2 + xy^2 + xyz$.
15. $3x^3y - 6x^2y^2 + 9x^2y^3$.
16. $4a^2x - 8a^2y - 6a^2b^2$.
17. $3a^4b^3 - 3a^5 - 15a^3b^2$.
18. $15a^2x - 10a^2y + 5a^2z$.
19. $a^{2n+4}b^3 - a^{3n+4}b^7$.
20. $\frac{4x}{y} - \frac{8x}{3y} - \frac{28x^3}{y^2}$.

WRITTEN EXERCISES

1–16. Write the other factor in Exercises 1–16 above.

II. POLYNOMIAL FACTORS

219. An expression may have a binomial or other polynomial factor that can readily be found by inspection.

For example:

1. $a + b$ is a factor of $(a + b)x + (a + b)y$.
2. And $x + y - z$ is a factor of $(x + y - z)ab - 3(x + y - z)cd$.

ORAL EXERCISES

State a factor of:

1. $(a+1)x-(a+1)y$.
2. $(a+x)x-(a+x)y$.
3. $a(b+c)x^2-a(b+c)y^2$.
4. $(m+n+p)ab+(m+n+p)cd$.
5. $(x+y)^2-(x+y)$.
6. $(a+1)^3+(a+1)^2+(a+1)$.

Supply the blanks:

7. $ax+ay+bx+by=(\quad)(x+y)+(\quad)(x+y)$
$=[(\quad)+(\quad)](x+y)$.
8. $ax+bx-ay-by=(\quad)(a+b)-(\quad)(a+b)$
$=[(\quad)-(\quad)](a+b)$.
9. $ax+bx+3a+3b=(\quad)(a+b)+(\quad)(a+b)$
$=[(\quad)+(\quad)](a+b)$.
10. $2ax^2-4ax+3x-6=(\quad)(x-2)+(\quad)(x-2)$
$=[(\quad)+(\quad)](x-2)$.
11. $6a+3b+9c+2ax+bx+3cx=(\quad)(2a+b+3c)$
$+(\quad)(2a+b+3c)=[(\quad)+(\quad)](2a+b+3c)$.

III. SQUARES OF BINOMIALS

220. Since $(x \pm y)^2 = x^2 \pm 2xy + y^2$, a trinomial is the square of a binomial, if one term is twice the product of the square roots of the other two, but not otherwise. Secs. 119, 120, p. 79.

For example:

$a^2 + 14a + 49 = a^2 + 2 \cdot 7a + 7^2 = (a + 7)^2$.

Here $14a$ is twice the product of $\sqrt{a^2}$ and $\sqrt{49}$.

$25m^2 - 30m + 9 = (5m)^2 - 2 \cdot 3 \cdot 5m + 3^2 = (5m - 3)^2$.

$16a^6 - 8a^3 + 1 = (4a^3)^2 - 8a^3 + (1)^2 = (4a^3 - 1)^2$.

Test by squaring.

WRITTEN EXERCISES

Factor:

1. $x^2+2ax+a^2$.
2. $x^2-2mx+m^2$.
3. $4x^2-4x+1$.
4. $9x^2-12x+4$.
5. $x^2y^2+2xy+1$.
6. $a^{2x}b^{2y}+2a^xb^yc+c^2$.
7. $a^2b^2+2abmn+m^2n^2$.
8. $a^2x^2-8ax+16$.
9. $(x+y)^2+2(x+y)+1$.
10. $\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}$.
11. $1+\frac{2}{x}+\frac{1}{x^2}$.
12. $(m-n)^2-2(m-n)x+x^2$.
13. $(a+b)^2-2(a+b)y+y^2$.
14. $9(p+1)^2-6(p+1)+1$.
15. $x^{2m}+4x^m+4+2(x^m+2)+1$.
16. $p^2q^2x^2y^2-6pqxy+9$.
17. $a^{2p}+144-24a^p$.
18. $p^{6n}+49-14p^{3n}$.
19. $49x^2+81y^2-126xy$.
20. $\frac{1}{m^4}+\frac{2}{m^2}+1$.
21. $\frac{x^2}{y^2}+2+\frac{y^2}{x^2}$.
22. $x^2+2xy+y^2+2(x+y)z+z^2$.

IV. THE DIFFERENCE OF TWO SQUARES

221. The difference of two squares is the product of the sum and the difference of the numbers whose squares occur.

Sec. 131, p. 83.

WRITTEN EXERCISES

Factor:

1. $a^2-b^2x^2$.
2. x^4-y^2.
3. $4a^2c-b^2c$.
4. $9x^2-16y^2$.
5. $25a^2b^2-1$.
6. $25a^2t^8-4b^2t^8$.
7. $49x^{2p+6}-1$.
8. $49a^2-4b^2$.
9. $1-a^2b^2c^2$.
10. $1-121x^2y^2$.
11. $1-x^4$.
12. $7a^2b^2-7c^2d^2$.
13. $16x^2y^2-4m^2n^2$.
14. ax^4-ay^4.
15. x^6-4y^6.
16. $c^5x^8-4c^5y^8$.
17. $81x^2y^2-9$.
18. $225a^4b^4-1$.
19. $25x^2y^4-36x^4y^2$.
20. $100a^2b^6-25a^6b^2$.
21. $\frac{a^2}{b^2}-1$.
22. $\frac{x^{2m}}{y^{2m}}-9$.
23. $16-\frac{p^{2n+4}}{q^{2n+4}}$.
24. $\frac{4x^2}{9y^2}-\frac{252^2}{49}$.
25. $\frac{16}{625x^2}-1$.

222. The terms of the given square may be polynomials, but the method of factoring is the same.

For example:

$$(a-b)^2-(b+c)^2=[(a-b)+(b+c)][(a-b)-(b+c)]$$
$$=(a-b+b+c)(a-b-b-c)$$
$$=(a+c)(a-2b-c).$$
$$(a^2+b)^2-(x^2+y)^2=(a^2+b+x^2+y)(a^2+b-x^2-y).$$

WRITTEN EXERCISES

Factor:

1. $(x+y)^2-(x-y)^2$.
2. $(a+b)^2-(a-b)^2$.
3. $(p+q)^2-(m+n)^2$.
4. $(a+b+c)^2-z^2$.
5. $(a-2b)^2-(3b+c)^2$.
6. $(a^2-1)^2-(b^2-1)^2$.
7. $a^2+2ab+b^2-c^2$.
8. $x^2-2xy+y^2-z^2$.
9. $4a^2-4a+1-9b^2$.
10. $16a^2b^2-a^2c^2-6ac-9$.
11. $p^2t^2-10pt+25-p^2+10pt-25t^2$.

V. FACTORING BY GROUPING

223. A factor of the given expression is sometimes seen more readily by first factoring groups of terms separately.

For example:

$$3x^3-5x^2+3x-5=x^2(3x-5)+3x-5=(3x-5)(x^2+1).$$

WRITTEN EXERCISES

Factor:

1. x^3+x^2+x+1.
2. x^3-2y-x^2y+2x.
3. $ax-ay+bx-by$.
4. $ax+3a+bx+3b$.
5. $a^2+ab-ac-bc$.
6. $ax+x-ay-y$.
7. $x^2-a^2+(x-a)^2$.
8. $5h^3-4h^2+10h-8$.
9. $6m^3+4m^2-9m-6$.
10. $xy-by-b+x$.
11. $x(z-a)^2-y(a-z)$.
12. $a(x+1)^2+3x+3$.
13. $4ax^2+bx-4ay^2-by$.
14. $4a^3+a^2-4a-1$.
15. x^3-4x^2+2x-8.
16. x^4-x^3-2x-2.
17. $(a+b+c)^2+ax+bx+cx$.
18. $(m^2+2mr+r^2)x^2+my+ry-m-r$.

Remove the parentheses and factor:

19. $2a^2 - x(x+a)$.

20. $y + x(x+y) - 1$.

21. $x(x+y) - a(a+y)$.

VI. COMPLETING THE SQUARE

224. It is often possible to factor an expression by first making it the difference of two squares. To do this a square is added to the expression, and to keep the value of the expression unaltered, the number added must also be subtracted.

For example:

$$a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = (a^2+b^2)^2 - a^2b^2$$
$$= (a^2 + b^2 + ab)(a^2 + b^2 - ab).$$

$$x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2+2)^2 - (2x)^2$$
$$= (x^2 + 2 - 2x)(x^2 + 2 + 2x).$$

$$16x^4 - x^2 + 1 = 16x^4 + 8x^2 + 1 - 9x^2 = (4x^2+1)^2 - 9x^2$$
$$= (4x^2+1)^2 - (3x)^2 = (4x^2 + 1 + 3x)(4x^2 + 1 - 3x).$$

TEST. $16 - 1 + 1 = 16 = (4 + 1 + 3)(4 + 1 - 3)$.

WRITTEN EXERCISES

Express as a difference of two squares and factor:

1. $a^4 + 4$.
2. $x^4y^4 + 4$.
3. $a^8 + 64$.
4. $64 + b^4$.
5. $x^4 + 4 \cdot 10^{8n}$.
6. $x^{4n} + 2x^{2n} + 9$.
7. $x^4 - 6x^2y^2 + y^4$.
8. $x^4 + 3x^2y^2 + 4y^4$.
9. $16a^4 + 4a^2 + 1$.
10. $4x^4y^4 + 3x^2y^2 + 1$.
11. $25x^4 + x^2 + 1$.
12. $x^4 + 4y^4$.
13. $81p^4 + 9p^2 + 1$.
14. $x^4 + 25x^2y^2 + 625y^4$.
15. $16a^4b^4 + 4a^2b^2 + 1$.
16. $81a^4 + 225a^2b^2 + 625b^4$.
17. $625x^4 + 400x^2y^2 + 256y^4$.
18. $a^4 + 2a^2b^2 + 9b^4$.
19. $x^4 - 8x^2y^2 + 4y^4$.
20. $4a^4 - 16a^2b^2 + 9b^4$.
21. $x^2 + 2xy - 15y^2$.
22. $a^2 + 6ab - 27b^2$.

VII. TYPE x^2+px+q

225. $(x+a)(x+b)=x^2+(a+b)x+ab$. Hence if a trinomial is the product of two factors like $x+a$ and $x+b$, the sum of a and b is the coefficient of the middle term, and their product, ab, is the third term.

The factors of such a trinomial are seen at once if a and b can be found by inspection.

For example:

In x^2+5x+6, $5=2+3$, and $6=2\cdot 3$.

$\therefore\ x^2+5x+6=x^2+(2+3)x+2\cdot 3=(x+2)(x+3)$.

In x^2-7x+6, $6=(-6)(-1)$ and $-7=-6+(-1)$.

$\therefore\ x^2-7x+6=x^2+(-6-1)x+(-6)(-1)=(x-6)(x-1)$.

In x^2+5x-6, $-6=6(-1)$ and $5=6+(-1)$.

$\therefore\ x^2+5x-6=x^2+(6-1)x+6(-1)=(x+6)(x-1)$.

ORAL EXERCISES

Factor:

1. a^2+3a+2.
2. d^2-3d+2.
3. a^2-5a+4.
4. x^2+5a+4.
5. y^2-6y+5.
6. m^2+6m+5.
7. x^2+8x+7.
8. x^2-8x+7.
9. a^2-a-2.

226. It may be helpful to write the various pairs of factors of the third term and then compare their sum with the coefficient of the middle term.

For example:

In $x^2-17x+72$, the pairs of factors of $+72$ are

72	36	24	18	12	9	
1	2	3	4	6	8	and the same pairs taken negatively.

Since the sum of the factors is negative, only the negative pairs need be examined, and by trial the pair -8, -9 is found to have the sum -17.

$$\therefore\ x^2-17x+72=(x-8)(x-9).$$

Likewise, in x^2-x-56, the factors of -56 are

-28	-14	-8	
2	4	7,	or the same numbers with the signs changed; but

since the coefficient of x is negative, only those pairs need be examined in which the negative number is the larger.

By trial, -8, 7 are found to have the sum -1.

$\therefore\ x^2-x-56=(x-8)(x+7)$.

Test by multiplication.

Factor: **WRITTEN EXERCISES**

1. x^2-x-30.	10. $x^2-7x-18$.	19. x^2+6x+5.
2. x^2+x-30.	11. $x^2+7x-18$.	20. $x^2+9x+20$.
3. x^2-x-20.	12. $x^2+17x+60$.	21. $x^2-8x+15$.
4. x^2+x-20.	13. $m^2+11m+28$.	22. x^2+8x+7.
5. $x^2-3x-18$.	14. $m^2-3m-28$.	23. $x^2-10x+9$.
6. $x^2+3x-18$.	15. $m^2+3m-28$.	24. $x^2+7x+12$.
7. a^2+a-42.	16. $y^2+6y-40$.	25. $x^2-5x-14$.
8. a^2-a-12.	17. $y^2-6y-40$.	26. $x^2+2x-15$.
9. $m^2-\frac{5m}{6}+\frac{1}{6}$.	18. $t^2-\frac{3t}{2}+\frac{1}{2}$.	27. $s^2-\frac{3s}{4}-\frac{1}{4}$.

28. $a^2b^2+11ab+24$.

29. $x^2y^2-20xy+100$.

30. $(a+b)^2-10(a+b)+9$.

31. $(m+n)^2+2(m+n)-15$.

VIII. TYPE mx^2+px+q

227. This type, whose factors have the form $(ax+b)(cx+d)$, can be reduced to the type x^2+px+q by multiplying and dividing the expression by m (the coefficient of x^2) and then putting $mx=y$. The method will be made sufficiently clear by an example.

EXAMPLE

Factor: $6x^2+19x+10$.

Multiplying and dividing the given trinomial by 6, we obtain

$$6x^2+19x+10=\tfrac{1}{6}(36x^2+6\cdot 19x+60).$$

Put $6x=y$.

Then, $36x^2+6\cdot 19x+60=y^2+19y+60$.

By Sec. 226,
$$\begin{aligned} y^2+19y+60 &= (y+4)(y+15) \\ &= (6x+4)(6x+15) \\ &= 2(3x+2)3(2x+5) \\ &= 6(3x+2)(2x+5). \end{aligned}$$

And finally, $6x^2+19x+10=\frac{1}{6}(y^2+19y+60)=(3x+2)(2x+5)$.

Test by multiplication.

WRITTEN EXERCISES

Factor:

1. $3x^2 + 7x + 2.$
2. $3x^2 - 5x - 2.$
3. $5x^2 - 9x - 2.$
4. $6a^2 - 8a - 8.$
5. $7p^2 + 22p + 3.$
6. $15z^2 + z - 6.$

The methods for factoring given in Sections 225, 226, and 227 are of limited value, for they determine the factors only in favorable instances. Thus, $x^2 - 17x + 72$ was readily factored in Section 226, but $x^2 - 16x + 72$ could not be so factored.

IX. TYPE $x^3 + y^3$ AND $x^3 - y^3$.

228. We know by multiplying that

$$(x + y)(x^2 - xy + y^2) = x^3 + y^3,$$

and that

$$(x - y)(x^2 + xy + y^2) = x^3 - y^3.$$

Hence,

One factor of the sum of two cubes is the sum of the numbers, and the other is the sum of the squares of the numbers minus their product.

And

One factor of the difference of two cubes is the difference between the numbers, and the other is the sum of their squares plus their product.

EXAMPLES

1. Factor: $27x^3 + 8y^3.$

$$27x^3 = (3x)^3.$$
$$8y^3 = (2y)^3.$$
$$\therefore 27x^3 + 8y^3 = (3x + 2y)\,[(3x)^2 - 3x \cdot 2y + (2y)^2]$$
$$= (3x + 2y)(9x^2 - 6xy + 4y^2).$$

Test by multiplication.

2. Factor: $8a^3b^3 - 125c^3.$

$$8a^3b^3 = (2ab)^3.$$
$$125c^3 = (5c)^3.$$
$$\therefore 8a^3b^3 - 125c^3 = (2ab - 5c)[(2ab)^2 + 2ab \cdot 5c + (5c)^2]$$
$$= (2ab - 5c)(4a^2b^2 + 10abc + 25c^2).$$

Test by letting $a = b = c = 1$: $8 - 125 = -3 \cdot 39.$

3. **Factor:** $8x^6 - a^6.$

$$8x^6 - a^6 = (2x^2)^3 - (a^2)^3$$
$$= (2x^2 - a^2)[(2x^2)^2 + (2x^2)(a^2) + (a^2)^2]$$
$$= (2x^2 - a^2)(4x^4 + 2a^2x^2 + a^4).$$

WRITTEN EXERCISES

Factor:

1. $a^3 - b^3.$
2. $a^3 - 8b^3.$
3. $8a^3 + b^3.$
4. $27x^3 - y^3.$
5. $y^3 + 27z^3.$
6. $a^{3n} + 1.$
7. $\frac{1}{a^3} - 1.$
8. $a^3 - 1.$
9. $x^6 - y^6.$
10. $a^6 + b^6.$
11. $64a^6 - 1.$
12. $8x^3 + 1.$
13. $27y^{6p} - 1.$
14. $a^3 - \frac{1}{27}.$
15. $xy^4 - x^4y.$
16. $125a^6 - b^3.$
17. $(a+b)^3 - 1.$
18. $27x^3y^3z^3 + 8.$
19. $a^3b^3 + (a+b)^3.$
20. $8m^3 - 27p^3q^3.$
21. $10^3 - h^{3n}.$
22. $(a+b)^3 - (b-c)^3.$
23. $(3a+b)^3 - (2a-b)^3.$
24. $(x+y)^3 - (x-y)^3.$
25. $(c+d)^3 + (2c-d)^3.$
26. $(x^3+1)^3 - (y^3+1)^3.$
27. $8a^3 + 125(b+c)^3.$

229. Types $x^3 - y^3$ and $x^3 + y^3$ may be used in calculation.

EXAMPLE

Calculate: $14^3 - 13^3.$

$$14^3 - 13^3 = (14 - 13)(14^2 + 14 \cdot 13 + 13^2)$$
$$= 14^2 + 14 \cdot 13 + 13^2$$
$$= 14(14 + 13) + 13^2$$
$$= 14 \cdot 27 + 13^2$$
$$= 378 + 169$$
$$= 547.$$

WRITTEN EXERCISES

Calculate similarly:

1. $16^3 - 15^3.$
2. $19^3 - 18^3.$
3. $23^3 - 21^3.$
4. $57^3 - 56^3.$

5. It is known that the volume of a sphere is $\frac{4\pi r^3}{3}$, r being the length of the radius. Using $\frac{22}{7}$ as an approximate value of π, calculate the number of cubic inches in a spherical shell whose outer radius is 14 in., and inner radius 13 in.

SOLUTION. The volume of the outer sphere is $\frac{4}{3}\cdot\frac{22}{7}\cdot 14^3$, and that of the inner sphere is $\frac{4}{3}\cdot\frac{22}{7}\cdot 13^3$.

Hence the volume of the shell in cubic inches is

$$\frac{4}{3}\cdot\frac{22}{7}(14^3-13^3)=\frac{4}{3}\cdot\frac{22}{7}\cdot 547 \qquad \text{(Sec. 229.)}$$
$$=2292^{+}$$

6. Find similarly the volumes of spherical shells if:

	(1)	(2)	(3)	(4)
Outer radius =	16	19	25	36
Inner radius =	15	18	23	32

FACTORING APPLIED TO FRACTIONS

230. We have already made considerable use of factoring in the treatment of fractions. The following miscellaneous exercises will furnish further practice.

WRITTEN EXERCISES

Reduce to lowest terms:

1. $\frac{x^2+2xy+y^2}{x^2-y^2}$.

2. $\frac{a^2-4}{a^2-4a+4}$.

3. $\frac{8ax+4ay-16az}{10bx+5by-20bz}$.

4. $\frac{x^3-1}{x^2-1}$.

5. $\frac{9x^2-12x+4}{9x^2-4}$.

6. $\frac{(a+b)^2-c^2}{ax+bx+cx}$.

7. $\frac{a^2-x^2+b^2+2ab}{a+b+x}$.

8. $\frac{x^4+3x^3+x+3}{2x+6}$.

Add:

9. $\frac{4}{x^2-y^2}+\frac{5}{x^3-y^3}$.

10. $\frac{x}{x^4-1}-\frac{x}{x^2+1}$.

11. $\dfrac{a}{a^4+a^2+1}+\dfrac{1}{a^2+a+1}$.

12. $\dfrac{x-3}{x^2-7x-18}+\dfrac{2x-1}{x^2-8x-9}$.

Multiply:

13. $\dfrac{x^3-8y^3}{x^2-y^2}\cdot\dfrac{x^2-xy-2y^2}{x^2-4xy+4y^2}$.

14. $\dfrac{4m+r}{25a^2-9}\cdot\dfrac{10am-6m}{16m^2+8mr+r^2}$.

15. $\dfrac{x^2+6x+5}{a^3+10a^2b+25ab^2}\cdot\dfrac{a^2-25ab}{x^2+2x+1}$.

16. $\dfrac{x^2+6ax+5a^2}{x^2+2ax+a^2}\cdot\dfrac{ab+bx}{5ac+cx}$.

Divide:

17. $\dfrac{a^2bc}{x^3-1}$ by $\dfrac{a^4bc^3}{x-1}$.

18. $\dfrac{a^2-16}{p+t}$ by $\dfrac{a^2-4a}{pt+t^2}$.

19. $\dfrac{a^2-3a-28}{x^4+4y^4}$ by $\dfrac{a^2-7a}{x^2+2xy+2y^2}$.

20. $\dfrac{a^4+10a^2x^2+25x^4}{5x^2}$ by $\dfrac{a^4+7a^2x^2+10x^4}{10x^2-35x^3}$.

FACTORING APPLIED TO EQUATIONS

231. Preparatory.

1. Find the values of the trinomial x^2-x-2 for $x=1$; 2; 0; -1; -2; 5; 4; 3.

2. For which values of x does the trinomial of Exercise 1 become zero?

3. Find the value of the binomial x^2-3x for $x=1$; -1; 2; -2; 0; 3; -3; 4; 5; 10.

4. For which values of x does the binomial of Exercise 3 become zero?

5. According to Exercises 1 and 2, what are the roots of the equation $x^2-x-2=0$?

6. According to Exercises 2 and 4, what are the roots of the equation $x^2 - 2x = 0$?

7. What roots are common to the two equations?

232. Equivalent Equations. If two equations have the same roots, the equations are said to be **equivalent.**

Thus,
$$4x = 12$$
and $$5x - 15 = 0$$
are equivalent, each having the root 3, and no others.

Also,
$$x^2 - 25 = 0$$
and $$4x^2 - 100 = 0$$
are equivalent, each having the roots 5, -5, and no others.

But,
$$x^2 - 25 = 0$$
and $$x^2 - 8x + 15 = 0$$
are not equivalent, for although the first has the roots 5, -5, and the second has the roots 5 and 3, the two equations have not the same roots.

WRITTEN EXERCISES

Write the roots of these equations and find which pairs are composed of equivalent equations:

1. $3x - 6 = 0$, $x - 3 = 4$.
2. $x - 3 = 5$, $2x = 16$.
3. $x^2 = 4$, $x + 2 = 0$.
4. $x^2 = 2$, $2x^2 = 4$.
5. $x - 4 = 8$, $x + 2 = 6$.
6. $x^2 = 9$, $x + 3 = 0$.

233. If two equations have between them the same roots as a third equation, the two together are said to be equivalent to the third.

EXAMPLES

1. One of the equations $x - 5 = 0$, and $x - 3 = 0$, has the root 5, the other the root 3. Between them, they have the roots 3 and 5, which are the roots of $x^2 - 8x + 15 = 0$. The equations $x - 3 = 0$ and $x - 5 = 0$, are together equivalent to $x^2 - 8x + 15 = 0$.

2. The equation $(x-1)(x-2) = 0$, asks: *For what values of x does the product $(x-1)(x-2)$ have the value zero?*

The product is zero, if either factor is zero, and not otherwise (Sec. 105, p. 67). $\therefore$ $(x-1)(x-2) = 0$, if $x - 1 = 0$, or if $x - 2 = 0$, and not otherwise.

Thus, the solution of the equation $(x-1)(x-2)=0$ depends upon the solution of $x-1=0$ and $x-2=0$. The roots of these being 1 and 2, the roots of $(x-1)(x-2)=0$, are likewise 1 and 2.

The pair of equations $x-1=0$, $x-2=0$ is equivalent to the equation $(x-1)(x-2)=0$.

ORAL EXERCISES

State the equations of the first degree that are equivalent to each of the following:

1. $(x-3)(x-2)=0.$
2. $(x-5)(x-3)=0.$
3. $(x-3)(x+2)=0.$
4. $(x+5)(x+3)=0.$
5. $(x-3)(x+10)=0.$
6. $x(x-5)=0.$
7. $(x+7)(x+1)=0.$
8. $x(x+3)=0.$
9. $x(x-a)=0.$
10. $(x+8)(x-11)=0.$

Factor the first member of each of the following equations and state the equations equivalent to each:

11. $x^2-3x+2=0.$
12. $x^2-5x+4=0.$
13. $x^2-6x+5=0.$
14. $x^2+6x+5=0.$
15. $x^2+8x+7=0.$
16. $x^2-4x-5=0.$
17. $x^2-9=0.$
18. $x^2-2x=0.$
19. $3x^2-2x=0.$
20. $x^2-14x+33=0.$

WRITTEN EXERCISES

1–10. Solve the equation in each exercise above from 11–20 by solving the equivalent equations.

Find the roots of each equation by factoring the left member and solving the equivalent equations:

11. $x^2-x-20=0.$
12. $x^2+x-30=0.$
13. $x^2-3x-18=0.$
14. $x^2-x-30=0.$
15. $x^2+3x-18=0.$
16. $x^2-x-42=0.$
17. $x^2-7x-18=0.$
18. $x^2-17x+72=0.$
19. $x^2-x-56=0.$
20. $x^2+7x-18=0.$
21. $x^2+11x+28=0.$
22. $x^2-3x-28=0.$
23. $x^2+6x-40=0.$
24. $x^2+11x+24=0.$

25. $x^2+3x-28=0$.
26. $x^2-6x-40=0$.
27. $x^2+18x+81=0$.
28. $x^2-20x+100=0$.
29. $x^2-8x+15=0$.
30. $x^2+9x+20=0$.
31. $x^2+8x+7=0$.
32. $x^2+2x-15=0$.
33. $x^2-x-6=0$.
34. $x^2-5x-14=0$.
35. $x^2+x-110=0$.
36. $x^2-5x-24=0$.

SUMMARY

I. **Definitions.**

1. Two equations are *equivalent*, if they have the same roots. Sec. 232.

2. A pair of equations is equivalent to a third equation, if the two equations together have the same roots as the third. Sec. 233.

II. **Properties and Processes.**

1. A factor of every term of a polynomial is a factor of the polynomial.

2. *Important Types:*

(1) $x^2 \pm 2xy + y^2 = (x \pm y)^2$.
(2) $x^2 - y^2 = (x+y)(x-y)$.
(3) $x^2+px+q=(x+a)(x+b)$, where $a+b=p$, and $ab=q$.
(4) $mx^2+px+q=(ax+b)(cx+d)$.
(5) $x^3+y^3=(x+y)(x^2-xy+y^2)$.
(6) $x^3-y^3=(x-y)(x^2+xy+y^2)$. Secs. 220–228.

3. Polynomials can sometimes be thrown into one of the above types by grouping. Sec. 223.

4. *Completing the Square:* If an expression can be made a square by adding a square, the expression is equivalent to the difference of two squares and can be factored by this type. Sec. 224.

5. Factoring may be used in solving equations. The equation

$$x^2-(a+b)x+ab=0, \text{ or } (x-a)(x-b)=0,$$

is equivalent to the pair of equations $x-a=0$, and $x-b=0$. Secs. 231–233.

REVIEW

ORAL EXERCISES

Factor:

1. $3a^2 - 15ab$.
2. $a^2 - x^2$.
3. $p^2 - 9q^2$.
4. $3a^4 - 6a^2b$.
5. $xy^3z - xyz^3$.
6. $16a^4 - 9c^4$.
7. $a^3 - x^3$.
8. $x^3 + z^3$.
9. $a^2x^4 - 2ax^2 + 1$.
10. $m^2 + 6m + 5$.

WRITTEN EXERCISES

Factor:

1. $x^2 - 2ax - 3a^2$.
2. $ac - ad + bc - bd$.
3. $-x^4 + x^2 + 12$.
4. $3a^2 + 12ab - 2a - 8b$.
5. $4a^2 - b^2 + 6a - 3b$.
6. $25x^4 - 10x^2y + y^2 - 9z^2$.
7. $16a^4 + 24a^2b + 9b^2$.
8. $25a^6 + 20a^3 + 4$.
9. $3x^2 - 7xy + 2y^2$.
10. $x^8 + 3x^4 - 4$.
11. $3a^2 + 6ab - 24b^2$.
12. $a^2 - 2ab - ac + 2bc$.
13. $m^4 - 11m^2n^2 + n^4$.
14. $a^2 - b^2 - (a - b)^2$.
15. $4x^4 + 1$.
16. $27x^2 + 3x - 2$.
17. $a(a + b) - c(c + b)$.
18. $2a^3 - 3a^2 - 2a + 3$.
19. $4x^4 + y^4 - 5x^2y^2$.
20. $x^2 - 4ax - 4b^2 + 8ab$.
21. $a^3x - a^2c + a^2by - ab^2x - b^3y + cb^2$.
22. $(a + b)(c^2 - d^2) - (a^2 - b^2)(c - d)$.
23. $a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$.
24. $a^2 + b^2 + 1 + 2b + 2a + 2ab - d^2$.
25. $x^{2n} - 2x^n + 1$.
26. $\frac{x^2}{9} - \frac{4y^2}{25}$.
27. $a^2 + \frac{5a}{6} + \frac{1}{6}$.

Solve by factoring:

28. $-x^2 + x + 12 = 0$.
29. $3x^2 - 7x + 2 = 0$.
30. $4z^2 + 12z + 9 = 0$.
31. $25x^2 + 20x + 4 = 0$.
32. $x^2 - x - 2 = 0$.
33. $x^2 - 64 = 0$.
34. $1 - x^2 = 0$.
35. $2y^2 + 12y + 10 = 0$.

Reduce to simplest forms:

36. $\dfrac{(a^2-b^2)x^2-2\,ax+1}{ax-bx-1}$.

37. $\dfrac{x^4-10\,x^2+9}{(x-1)(x-3)}$.

38. $\dfrac{a^3-2\,a^2-a+2}{a^2-1}$.

39. $\dfrac{y^4-5\,y^2+4}{y^2+y-2}$.

40. By factoring the expressions find the highest common factor of $x^4+x^2y^2+y^4$ and x^3+y^3.

41. By factoring the expressions find the lowest common multiple of $x^2-3\,x+2$, x^2-1, and $x^2+2\,x+1$.

42. By factoring the expressions find the highest common factor of $(2\,x-1)(x^3-1)$ and $(x^3+x^2+x)(x-1)(x^2-1)$.

43. By factoring the expressions find the highest common factor and the lowest common multiple of the two expressions:

$$(x^2-1)(x^2+5\,x+6),\ (x^2+3\,x)(x^2-x-6).$$

SUPPLEMENTARY WORK

ADDITIONAL EXERCISES

Factor by reference to:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$$

1. $m^2 + q^2 + r^2 + 2\,mq + 2\,mr + 2\,qr.$
2. $a^2 + b^2 + x^2 + 2\,ab - 2\,ax - 2\,bx.$
3. $x^2 + y^2 + 25 + 2\,xy + 10\,x + 10\,y.$
4. $9 + a^2 + m^2 + 2\,am - 6\,a - 6\,m.$
5. $t^2 + 2\,tv + v^2 - 3\,t + 9 - 3\,v.$
6. $2\,ag + 16 - 8\,g + a^2 - 8\,a + g^2.$
7. $25\,x^2 + 9\,y^2 + 4\,z^2 + 30\,xy - 20\,xz - 12\,yz.$
8. Factor:

$$x^4 - 2\,x^2a^2 - 2\,x^2b^2 + a^4 + b^4 + 2\,a^2b^2.$$

9. Resolve into four factors:

$$x^4 + y^4 + z^4 - 2\,x^2y^2 - 2\,x^2z^2 - 2\,y^2z^2.$$

SUGGESTION. By adding and subtracting $4\,y^2z^2$ the expression reduces to $(x^2 - y^2 - z^2)^2 - 4\,y^2z^2$.

Highest Common Factor by Division

Heretofore we have found the highest common factor of two or more algebraic expressions by factoring them and selecting the common factors. A more general method may be seen in the long division process of finding the g. c. d. of two arithmetical numbers.

Thus, find the greatest common divisor of 1833 and 3760.

1. Divide the larger number by the smaller:

$$\begin{array}{r} 2 \\ 1833\overline{)3760} \\ \underline{3666} \\ 94 \end{array}$$

Hence, $3760 = 2 \times 1833 + 94$
or $3760 - 2 \times 1833 = 94$.

2. Since any factor of both terms of a binomial is a factor of the binomial, any factor of 3760 and 1833 is a factor of $3760 - 2 \times 1833$; that is, of 94.

The g. c. d. is therefore a factor of 94; and since $3760 = 2 \times 1833 + 94$, any common factor of 94 and 1833 is a factor of 3760. Therefore the g. c. d. of 1833 and 3760 is the g. c. d. of 94 and 1833.

3. Divide as in (1):

$$\begin{array}{r} 19 \\ 94\overline{)1833} \\ \underline{94} \\ 893 \\ \underline{846} \\ 47 \end{array}$$

Hence, $1833 = 19 \times 94 + 47$.

By the same reasoning as in step (2) the g. c. d. of 1833 and 94 is the g. c. d. of 94 and 47.

4. Divide as before:

$$\begin{array}{r} 2 \\ 47\overline{)94} \\ \underline{94} \end{array}$$

Hence, $94 = 2 \times 47$.

That is, 94 is a multiple of 47, and 47 is the g. c. d. of 47 and 94, therefore of 1833 and 3760.

The process is as follows: Divide the larger number by the smaller. If the remainder zero is reached, the divisor is the g. c. d. of the two given numbers. If the remainder is not zero, use it as divisor, and the preceding divisor as dividend. Repeat this process until the remainder zero is reached. The divisor for which this occurs is the g. c. d.

H. C. F. of Polynomials

The method given above can be used to determine the h. c. f. of two polynomials. In the following proof the two polynomials are supposed to involve only one letter.

Let A and B denote the given polynomials, the degree of B being not greater than that of A.

Divide A by B, continuing the division until a remainder R_1 is reached of degree less than that of B. Let Q_1 denote the quotient. Then:

$$A = Q_1B + R_1, \qquad (1)$$

or

$$A - Q_1B = R_1. \qquad (2)$$

Any common factor of A and B is a factor of the left member of (2) and hence of R_1 and (from 1), a common factor of B and R_1 is a factor of A. Therefore the h. c. f. of A and B is the h. c. f. of B and R_1. Thus the problem is reduced to that of finding the h. c. f. of B and R_1.

Proceed as before and divide B by R_1. Then $B = Q_2R_1 + R_2$, where R_2 is of lower degree than R_1. As above, the h. c. f. of R_1 and R_2 is the h. c. f. of B and R_1; the latter was seen to equal the h. c. f. of A and B. Thus the problem of finding the h. c. f. of A and B has been reduced to that of finding the h. c. f. of R_1 and R_2. By continuing this process, the problem is reduced successively to that of finding the h. c. f. of other pairs of expressions of lower degrees. Since in each division the degrees of both the dividend and divisor are lower than in the preceding, the process must ultimately come to an end.

Suppose that $$R_k = Q_{k+2}R_{k+1}.$$

Then since R_{k+1} is a factor of R_k, it is the h. c. f. of R_k, R_{k+1}, and hence of the next previous pair, and so on, back to A and B.

In the most unfavorable case the process of division does not come to an end until the dividend is of the first degree and the divisor an arithmetical number. In this case the two expressions have no literal common factor. Any numerical common factor can be seen by inspection at the outset.

EXAMPLES

1. Find h. c. f., $6x^4 + 7x^3 - 4x^2 - 1$ and $6x^3 + 7x^2 - 5x - 1$.

Dividing,

$$\begin{array}{r|l} & x \\ 6x^3 + 7x^2 - 5x - 1 & 6x^4 + 7x^3 - 4x^2 - 1 \\ & \underline{6x^4 + 7x^3 - 5x^2 - x\phantom{{}-1}} \\ & \phantom{6x^4 + 7x^3 - {}} x^2 + x - 1 \end{array}$$

$$\begin{array}{r|l} & 6x + 1 \\ x^2 + x - 1 & 6x^3 + 7x^2 - 5x - 1 \\ & \underline{6x^3 + 6x^2 - 6x\phantom{{}-1}} \\ & \phantom{6x^3 + {}} x^2 + x - 1 \\ & \phantom{6x^3 + {}} \underline{x^2 + x - 1} \end{array}$$

The h. c. f. is $x^2 + x - 1$.

2. Find h. c. f., $30x^3 - 76x^2 + 20x + 35$ and $10x^2 - 12x - 11$.

The division may be performed with detached coefficients (p. 108).

$$\begin{array}{r|l} & 3 - 4 \\ 10 - 12 - 11 & 30 - 76 + 20 + 35 \\ & \underline{30 - 36 - 33\phantom{{}+35}} \\ & - 40 + 53 + 35 \\ & \underline{- 40 + 48 + 44} \\ & \phantom{30 - 40 + {}} 5 - 9 \end{array}$$

$$
\begin{array}{r|l}
 & \quad 2 + \frac{6}{5} \\
5 - 9) & 10 - 12 - 11 \\
 & \underline{10 - 18} \\
 & \qquad 6 - 11 \\
 & \qquad \underline{6 - \frac{54}{5}} \\
 & \qquad\quad - \frac{1}{5}
\end{array}
$$

Since the remainder is numerical, the two expressions have no literal common factor. By inspection it appears that they have no numerical common factor.

To find the h. c. f. of more than two polynomials, find the h. c. f. of two of them, then the h. c. f. of this result and a third one of the polynomials, and so on.

WRITTEN EXERCISES

Find the h. c. f. of:

1. $2x^4 + 8x^3 - 19x^2 - 53x - 10$ and $2x^3 + 4x^2 - 29x - 5$.
2. $x^3 + 4x^2 - x - 12$ and $x^5 + 4x^4 - 2x^3 - 15x^2 + 2x + 8$.
3. $6x^4 + 19x^3 + 35x^2 + 10x - 1$ and $2x^3 + 5x^2 + 8x - 3$.
4. $16x^6 + 6x^5 + 15x^3 - 6x^2 - 1$ and $2x^5 + x^4 - 2x^2 - x$.
5. $12x^4 + 20x^3 - 27x^2 - 27x + 27$ and $6x^3 + 13x^2 - 8x - 21$.

To find the Lowest Common Multiple by use of the H. C. F.

The l. c. m. of two expressions is the result of dividing their product by their h. c. f. For:

1. **Let A and B be the given expressions and h their h. c. f.**
2. **Then $A = hm$ and $B = hn$, where m and n have no common factors.**
3. **$\frac{AB}{h} = \frac{hm \cdot hn}{h} = hmn$, dividing the product by the h. c. f.**
4. **But hmn contains all of the factors of A and B without the repetition of any common factor, hence it is the l. c. m. of A and B.**

The l. c. m. is most readily found by dividing one of the given expressions by the h. c. f. and multiplying the other expression by this quotient.

EXAMPLES

1. Find the l. c. m. of 17,273 and 564,001.

1. **The g. c. d. of these numbers is 751.**
2. **$17{,}273 \div 751 = 23$.**
3. **$\therefore$ the l. c. m. $= 23 \times 564{,}001 = 12{,}972{,}023$.**

2. Find the l. c. m. of $x^3 - 9x^2 + 23x - 15$ and $x^2 - 8x + 7$.

1. The h. c. f of these numbers is $x - 1$.
2. $(x^2 - 8x + 7) \div (x - 1) = x - 7$.
3. $\therefore$ the l. c. m. $= (x - 7)(x^3 - 9x^2 + 23x - 15)$.

WRITTEN EXERCISES

Find the l. c. m. of:

1. $3a^2 - 5a + 2$ and $4a^3 - 4a^2 - a + 1$.
2. $x^3 + 2x^2y - xy^2 - 2y^3$ and $x^3 - 2x^2y - xy^2 + 2y^3$.
3. $m^3 - 10m^2 + 31m - 30$ and $m^3 - 11m^2 + 38m - 40$.
4. $a^4 - 10a^2 + 9$ and $a^4 + 4a^3 - 22a^2 - 4a + 21$.
5. $2x^3 + (2a - 3b)x^2 - (2b^2 + 3ab)x + 3b^2$ and $2x^2 - (3b - 2c)x - 3bc$.

CHAPTER XIV

EQUATIONS

DEFINITIONS AND PROPERTIES

234. The following is a summary and extension of the preceding treatment of equations.

1. When two expressions are equal the statement of this equality by use of the symbol $=$ is called an **equation.**

2. Such values of the letters as make two expressions equal are said to **satisfy** the equation between these expressions. Sec. 16, p. 10.

3. Equations that are satisfied by any set of values whatsoever for the letters involved are called **identities.** Sec. 11, p. 9.

4. Equations that are satisfied by particular values only are called **conditional equations,** or, when there is no danger of confusion, simply **equations.** Sec. 12, p. 9.

5. The numbers that satisfy an equation are called the **roots** of the equation. Sec. 17, p. 10.

6. To **solve** an equation is to find its roots. Sec. 18, p. 10.

7. The letters whose values are regarded as unknown are called the **unknowns.**

Unless otherwise specified the later letters of the alphabet are understood to be unknowns, and the earlier letters to represent known numbers.

235. If two numbers are equal, the numbers are equal which result from:

1. *Adding the same numbers to each.* Sec. 20, p. 11.

2. *Multiplying each by the same number.* Sec. 20, p. 11.

Subtraction of the same number, and division by the same number not zero are here included under addition and multiplication; for to subtract a number is to add its negative, and to divide by a number is to multiply by its reciprocal.

236. *Transposing Terms.* If the opposite of any term occurring in an equation is added to both members of the equation, the term will be neutralized where it stands, but its opposite will appear in the other member. In other words, the balance of an equation is not disturbed if a term is removed from one member and inserted with its sign changed in the other. This is called **transposing.**

EXAMPLE

Solve: $$4x+5=6-3x. \qquad (1)$$

Adding $3x$ to both members, $$4x+5+3x=6-3x+3x, \qquad (2)$$

or, $$4x+5+3x=6. \qquad (3)$$

The effect is to remove the term $-3x$ from the right member and to insert it with its sign changed in the left member.

Adding -5 to both members of (3), $$4x+5+3x-5=6-5, \qquad (4)$$

$$4x+3x=6-5. \qquad (5)$$

The effect is to remove 5 from the left member and to insert it with its sign changed in the right member.

237. *Clearing of Fractions.* If both members of an equation involving fractions are multiplied by the l. c. m. of the denominators, the resulting equation when simplified will be free from fractions. This is called **clearing of fractions.**

EXAMPLE

Solve: $$\frac{5}{2x}+\frac{x}{3y}=\frac{7}{4}. \qquad (1)$$

Multiplying both members of (1) by $12xy$, the l. c. m. of the denominators, $$\frac{5(12xy)}{2x}+\frac{x(12xy)}{3y}=\frac{7(12xy)}{4}. \qquad (2)$$

Simplifying the fractions in (2), $$30y+4x^2=21xy. \qquad (3)$$

Whenever it is possible to make the calculations mentally without error, the equation corresponding to (2) should not be written.

238. Testing. When a root is substituted in both members of an equation, the results are equal numbers. This test should always be applied to the numbers found by solving. Sec. 22, p. 11.

239. Degree of an Equation.

1. The degree of an equation is stated with respect to its unknowns. It is the highest degree to which the unknowns occur in any term in the equation. Unless otherwise stated, all the unknowns are considered.

For example:

1. $3x+1=0$, and $4x+7y-3z=6$, are equations of the first degree.

2. $4y^2-y+3=0$, and $x^2+y^2-4=0$, are equations of the second degree.

3. $x^3-1=0$, $x^3+2y^2-4x+3=0$, $5xyz+x^2=2y$, and $x^3-3y^3+z=5x$, are equations of the third degree.

4. $v=\frac{4\pi R^3}{3}$ is of the first degree in v and of the third degree in R.

2. An equation of the first degree is called a **linear equation.**

3. An equation of the second degree is called a **quadratic equation.**

4. An equation of the third or higher degree is called a **higher equation.**

It is unnecessary to define here the degree of expressions containing radicals or fractions.

5. In order to state the degree of an equation its terms must be united as much as possible.

Thus, $x^2+2x+1=x^2$ appears to be a quadratic equation.
But $2x+1=0$, to which it reduces, is a linear equation.

6. Terms not involving the unknowns are called **absolute terms.**

Thus, in $x^2+5=3x-2a$, 5 and $-2a$ are absolute terms.

ORAL EXERCISES

State the degree of each of the following equations with respect to each unknown:

1. $x+2=0$.
2. $ax+2=0$.
3. $z^2-mz=4$.
4. $x^2+xy=5$.

5. $x^2 - 4 = 0.$

6. $\frac{1}{2}mv^2 = 6.$

7. $gt^2 = 32.$

8. $at + \frac{1}{2}gt^2 = 0.$

9. $ax^2 + b = 0.$

10. $x^2 - xy = 2.$

11. $ax^3 - by^2 = 1.$

12. $a^2x^2 + b^2y^2 = a^2b^2.$

13. $\frac{4}{3}\pi r^3 = 100.$

14. $\dfrac{w^2 - 1}{4} = 6.$

15. $2\pi r^2 h = 50.$

16. $mv^2 - 16 = 0.$

17. $t_1^2 - t_2^2 = 3t_1t_2.$

18. $xy^2 - x^2y = 6.$

Select the linear, the quadratic, and the higher equations from the following:

19. $x + 3 = 0.$

20. $x^2 = 9.$

21. $x^3 = 27.$

22. $2x - 6 = 0.$

23. $ax = b + c.$

24 $mx^2 + px = q.$

25. $x^3 - x^2 - x = 3.$

26. $ax^2 + ax = c.$

27. $2x - 8x = 0.$

28. $3x^2 - x^2 + 3x = x^2 + 1.$

29. $2x^3 = 2x^2 + x + 2x^3.$

30. $x^4 + 5 = 3.$

31. $ay^3 - by^2 + 1 = 0.$

32. $5y^2 + 2y = 6 + 5y^2.$

240. Equivalent Equations. If two equations have the same roots, the equations are said to be **equivalent**. If two equations have together the same roots as a third equation, the two equations together are said to be equivalent to the third.

SOLUTION OF EQUATIONS

241. To Solve a Linear Equation with One Unknown. In general:

1. *Clear it of fractions if there are any.*
2. *Remove the parentheses if there are any.*
3. *Transpose all the terms containing the unknown quantity to one member, preferably the left, and all the other terms to the other member.*
4. *Unite the terms in each member as much as possible.*
5. *Divide both members by the coefficient of the unknown.*

EXAMPLE

Solve: $$\frac{2x-5}{6}+\frac{6x+3}{4}=5x-\frac{35}{2}. \quad (1)$$

Multiplying each term by 12, the l. c. m. of the denominators, $2(2x-5)+3(6x+3)=60x-6\cdot 35. \quad (2)$

Removing the parentheses, $4x-10+18x+9=60x-210. \quad (3)$

Transposing -10, 9, and $60x$, $4x+18x-60x=10-9-210. \quad (4)$

Uniting terms, and multiplying both members by -1, $38x=209. \quad (5)$

Dividing both members by 38, $x=\frac{11}{2}. \quad (6)$

TEST. $$\frac{2(\frac{11}{2})-5}{6}+\frac{6(\frac{11}{2})+3}{4}=5(\tfrac{11}{2})-\frac{35}{2}.$$

WRITTEN EXERCISES

Solve and test:

1. $2(x+3)-3(x+2)=6(x+5)$.

2. $x(x-1)-(2x-1)=x(x+6)$.

3. $2x-\frac{11+x}{2}=\frac{19+x}{3}$.

4. $(x+3)^2=-7+(5-x)^2$.

5. $\frac{z-1}{2}+\frac{z-2}{3}-6=\frac{z-3}{4}$.

6. $\frac{x}{4}+\frac{4x-12}{5}-\frac{9x+2}{20}=0$.

7. $\frac{y}{12}-\frac{8-y}{8}+2\frac{3}{4}=\frac{5+y}{4}$.

8. $5(x-4)+4(x-3)-x(x-1)=x(3-x)$.

9. $\frac{1}{2}(x-5)-\frac{1}{3}(x-4)=\frac{1}{2}(x-3)-(x-2)$.

242. Formula. An expression which shows how a desired number is to be obtained from given numbers is called a **formula.**

The following examples show how formulas arise in the solution of problems:

EXAMPLES

1. The sum of three numbers is 40. The second is 6 more than the first, and the third is the sum of the other two. Find the numbers.

SOLUTION. 1. Let $x =$ the first number.

2. Then $x + 6 =$ the second number,

and $x + x + 6 =$ the third number.

3. Therefore, $x + x + 6 + x + x + 6 =$ sum,

or
$$4x + 12 = 40.$$
$$4x = 28.$$
$$x = 7.$$

4. Therefore the numbers are 7, 13, and 20.

TEST. $7 + 13 + 20 = 40$; $13 = 7 + 6$; $20 = 13 + 7$.

In this work the given numbers have quite disappeared in the course of the operations leading to the result. Nothing shows how the resulting numbers 7, 13, and 20 are connected with the given numbers 6 and 40. This connection is made clear by solving the following problem:

2. The sum of three numbers is s; the second is a greater than the first, and the third is the sum of the other two. Find each number.

SOLUTION. 1. Let $x =$ the first number.

2. Then $x + a =$ the second number.

3. And $x + x + a =$ the third number.

4. $\therefore x + x + a + x + x + a =$ the sum,

or
$$4x + 2a = s.$$
$$4x = s - 2a.$$
$$x = \frac{s - 2a}{4}.$$

5. $$\therefore x + a = \frac{s - 2a}{4} + a = \frac{s - 2a + 4a}{4} = \frac{s + 2a}{4},$$

6. and $$x + x + a = \frac{s - 2a}{4} + \frac{s + 2a}{4} = \frac{s}{2}.$$

7. $\therefore$ the numbers are $\frac{s - 2a}{4}$, $\frac{s + 2a}{4}$, $\frac{s}{2}$.

That is, the result is connected with the given numbers as follows: The first number is *one fourth of the difference between the given sum and twice the given excess of the second number over the first.* This enables us at once to write out the answer for *any* such problem. In the first problem above s is 40, a is 6, hence the first number is $\frac{40 - 12}{4}$, or 7. The other numbers may be found similarly.

When the given numbers are indicated by letters, we are dealing with a family of problems rather than with a single problem. The result is a *formula*, as in step 7 above, not a numerical solution.

A formula shows the **form** of a result and contains the solution of all problems which may be made by simply changing the given numbers in any problem.

243. Form. The nature of mathematical forms is well illustrated by a blank check. This is a **form** for all checks. When the various blanks are filled out, it becomes a particular check. But the characteristics common to all checks can be seen fully in the blank form.

Similarly, in a polynomial in which letters are used to represent unspecified numbers, we have in reality a blank form for a certain family of polynomials. When the blanks indicated by letters are filled out by specific numerical values, the result is a particular polynomial.

244. The linear form, $ax + b$. Every polynomial of the first degree can be put into the form $ax + b$. That is, by rearranging the terms suitably, it can be written as *the product of x by a number not involving x, plus an absolute term.* Hence, the form $ax + b$ is called a *general form* for all polynomials of the first degree in x.

Thus, $4x + 3(2 - 5x)$ can be written $-11x + 6$, which is in the form of $ax + b$, because -11 takes the place of a and $+6$ takes the place of b.

WRITTEN EXERCISES

Put the following polynomials into the form $ax + b$:

1. $3x + 7x - 5$.
2. $8(x-1) + 4(x-3)$.
3. $6x(1+x) + 3(1-2x^2)$.
4. $\frac{3}{5}(x - \frac{1}{6}) + \frac{2}{15}(x + \frac{1}{4})$.
5. $5q + 2q(x+7)$.
6. $mx + c(x+1)$.
7. $(1-x)(5+x)-(3+x)(6-x)$.
8. $2.3x - 0.6(3x + 1.4)$.
9. $\dfrac{2x-3}{8} - \left(\dfrac{5x}{4} + \dfrac{3x+1}{16}\right)$.
10. $400 - \dfrac{x}{6} + 399\frac{2}{3} - \dfrac{3x+5}{12} - \dfrac{9x+1}{2}$.

11. $(x^2-1)^2-(x^2+3)^2+8x^2-2(x+1)+8.$

12. $(4x+3)(2x-9)+8x(2-x).$

13. $(x+1)^3-(x-1)^3-6x^2+2x.$

14. $(40x-1)^2-(41x+2)^2+(9x+3)^2.$

15. $(17x-4)^2-2(12x+3)^2-(x-1)^2.$

16. $(3x+1)(4x-1)+5(x+2)^2-17(x^2-1).$

17. What is the 5th even number? The 10th? The 27th? The ath?

18. What is the 4th odd number? The 10th? The 40th? The kth?

If n represents an integer:

19. Show that $2n$ is a general form for all even numbers.

20. Show that $2n+1$ is a general form for all odd numbers.

21. What is the 5th multiple of 3? The 18th? The rth?

22. Show that $3n$ is a general form for all integral multiples of 3.

245. General Equations. Just as every linear polynomial is of the general form $ax+b$, so every equation of the first degree is equivalent to an equation of the form

$$ax+b=0;$$

consequently the latter is called *a general equation of the first degree with one unknown.*

246. General Solution. From the equation $ax+b=0$, (1)

we have, $$ax=-b, \quad (2)$$

and hence, $$x=\frac{-b}{a}. \quad (3)$$

247. $\frac{-b}{a}$ is the *general form of the root* of the equation of the first degree. There is always one *root*, and only one.

The advantage of a general solution like this is that it leads to a *formula* which is applicable to all equations of the given form.

248. The rule for the solution of equations of first degree is: *Put the equation into the form $ax + b = 0$. The root will be the negative of the term not involving x divided by the coefficient of x.*

WRITTEN EXERCISES

Simplify and solve according to the above rule:

1. $1 = 4x(3 + 7)$.
2. $6 + 8(1 - x) = 2$.
3. $5x + 3(x - 2) = 4x$.
4. $a = bx + c$.
5. $\frac{1}{x} = \frac{3}{4}$.
6. $\frac{x+1}{x+2} = 7$.
7. $\frac{x+1}{x} = \frac{4}{7}$.
8. $\frac{x}{1+x} = 2$.
9. $a = \frac{4y+5}{7-y}$.
10. $3px + 4bd = 7ax - b + d$.
11. $w(1 - a) + a(w - 1) = 0$.
12. $5z + 15 + \frac{3z+16}{5} = \frac{6z+91}{5}$.
13. $3(x+1) + 7(x+2) = 4(x+3)$.
14. $(x - a)(x - b) = x^2 - a$.
15. $3bx - 7(x + b) + ac - cx = 0$.
16. $4 - \frac{5-x}{2} = \frac{3a+x}{4}$.
17. $2[3(x + 1) + 1] + 1 = 0$.
18. $\frac{1}{x+1} - \frac{a}{x+a} = 0$.
19. $(3 - x)(7 - x) = 10 + x^2$.
20. $x(1 + a) + 2x(1 + b) = 3x(1 + c)$.
21. $(1 + 3x)^2 + (3 + 4x)^2 = (1 - 5x)^2$.
22. $(1 - x)(4 + x) + (2 + x)(6 + x) = 0$.
23. $(a - x)(b + x) = c^2 - x^2$.
24. $(ax + b)(cx + d) = ac(x + 1)^2$.

Solve for x:

25. $\frac{(m+1)x}{x-1} - \frac{x}{x+1} = m$.
26. $\frac{1}{a+b} + \frac{a+b}{x} = \frac{1}{a-b} + \frac{a-b}{x}$.

Solve for a:

27. $ab + ac = 4$.
28. $ag + 3 = 2ah - 5$.
29. $(a - 1)(a + 2) = (a - b)(a + b)$.
30. Find the value of e in the equation:

$$\frac{ab}{e} = bc + d + \frac{1}{e}.$$

31. Solve the preceding equation for b.

Solve for g:

32. $v = at + \dfrac{gt^2}{2}$. **33.** $gh + gh^2 = r^3$.

34. A man doing physical labor should eat daily a certain weight of starchy foods, 16 % of that weight of fats, and 20 % of that weight of albuminous food (protein). The total weight of these three foods consumed daily should average at least $1\frac{1}{2}$ lb. What average weight of each is required?

35. 18 % by weight of wheat is lost (as bran, etc.) in grinding it into flour. How many 60-pound bushels of wheat are used in making 738 lb. of flour? In making p lb.?

36. The weight of bread is $133\frac{1}{3}$ % of the weight of the flour used to make it. According to Exercise 35 how many one-pound loaves of bread can be made from 20 bu. of wheat? From b bu.?

37. In a recent year, there were 28 cities of population over 30,000 in the states of New York, Illinois, Iowa, and Texas together. The number in Iowa was $\frac{5}{4}$ that in Texas. The number in Illinois was 2 more than that in Iowa, and the number in New York was equal to that in Iowa and Illinois together. Find the number in each state.

38. In the cities of the United States numbered in order of population (1900), the number of Milwaukee is $\frac{7}{5}$ that of Cincinnati, and $\frac{1}{2}$ that of Columbus, Ohio. The sum of the three numbers is 52. Find the number of each city.

39. Divide the number n into four parts A, B, C, D, such that A shall be $\frac{3}{2}$ of B, and $\frac{4}{3}$ of $(B + C)$, and $\frac{5}{4}$ of $(A + C)$.

40. Apply the results of Exercise 39 to solve the same problem when $n = 149$.

41. A commission merchant sells goods for s dollars and remits r dollars; what is the rate of his commission?

42. Given $a =$ the amount, $r =$ the rate, and $t =$ the time in a problem of simple interest, what is the principal?

43. A train leaves A for a station B, d miles distant; its rate is r miles per hour; h hours later another train leaves B for A, running r' miles per hour. How far will each train have traveled when they meet?

SOLUTION. 1. Let $x =$ the number of hours traveled by the first train before they meet.

2. Then, $x - h =$ the number of hours traveled by the second train.
3. $rx =$ number of miles traveled by first train.
4. $r'(x - h) =$ number of miles traveled by second train.
5. $\therefore rx + r'(x - h) = d$, by the conditions of the problem.
6. $\therefore (r + r')x = d + r'h$, performing the operations.
7. $\therefore x = \dfrac{d + r'h}{r + r'}$, solving (6).
8. $\therefore r = \dfrac{r(d + r'h)}{r + r'}$, the distance traveled by the first train.
9. And $r'(x - h) = \dfrac{r'(d - rh)}{r + r'}$, the distance traveled by the second train.

44. Read the above problem, taking $d = 450$, $r = 50$, and $r' = 40$, and $h = 4\frac{1}{2}$. Solve the problem by substituting these numbers in the formulas of steps 8 and 9.

45. How much water must be added to a 20% solution of ammonia to make a 10% solution?

PLAN. 1. Consider an arbitrary quantity of the given mixture; for example, 1 gallon.

2. Let x be the number of gallons added; then the two quantities are:

1ST QUANTITY	2D QUANTITY
1 gal.	$(1 + x)$ gal.

3. Decide which substance (the ammonia in this problem) has not changed in quantity; state the amount in each solution. Thus,

AMMONIA IN 1ST SOLUTION	AMMONIA IN 2D SOLUTION
20% of 1 gal.	10% of $(1 + x)$ gal.

4. Therefore, the equation is 20% of $1 = 10\%$ of $(1 + x)$, or $.2 = .1(1 + x)$.

5. Therefore, $x = 1$, and 1 gallon of water must be added for each gallon of the original solution.

Solve the following, using the tabular plan given above:

46. How much water must be added to a 95% solution of alcohol to make an 80% solution?

47. A certain paint consists of equal parts of oil and pigment. How much oil must be added to a gallon of this paint to make a paint $\frac{2}{3}$ of which is oil?

48. How much potash must be added to a 10% solution to make a 20% solution?

49. Spirits of camphor is camphor gum dissolved in alcohol. How many ounces of camphor gum must be added per ounce to a 5% solution in order to make an 8% solution?

GRAPHICAL WORK

249. PREPARATORY.

1. The average daily expenses of a factory, for gas, rent, etc., is \$10. The daily profit per workman above wages, cost of material, etc., is 50¢ (.5 of a dollar). What would be the daily loss, if only one man were employed? 2 men? 3 men?

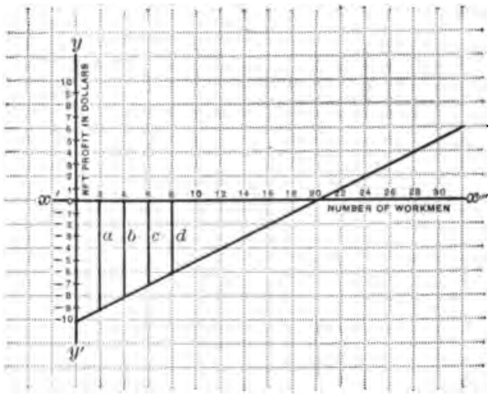
Letting x represent the number of men, $.5x$ will represent the daily profit of x men. Thus, the answers to the above questions may be computed by substituting the number of men for x in the expression $.5x - 10$.

2. In the language of algebra what kind of net profit would this loss be called?

3. How many men must be employed so that the daily profit shall equal the expenses, or so that the net profit shall be zero? (Use $.5x - 10$ to find the answer.)

4. How many men must be employed so that the net profit shall be \$5? \$10? \$20? \$25? (Use $.5x - 10$ to find the answers.)

5. In the diagram, what kind of net profit do the lines, *a, b, c, d,* represent? These are the net profits from 2, 4, 6, 8 men respectively. Read these profits in dollars.



6. *a, b, c, d* represent values of what expression?

7. For how many men do the net profits become zero? How does the graph show this?

8. Read from the graph the net profits for 24 men; 28 men; 30 men.

9. This graph represents the change in value of what expression?

WRITTEN EXERCISES

1. Represent graphically the net profits from 30 to 50 men on the same basis as given above. Read from the graph the net profits from 45 men; 35 men; 40 men.

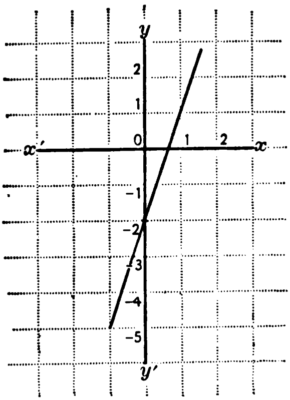
2. Represent graphically the net profits from 1 to 50 men when the fixed daily expenses are \$25 and the gross profit per man is \$1. What is the least number of men the factory can employ without loss? What is the profit from 35 men? From 50 men?

3. The profits of a certain factory are 50¢ daily for each workman employed, less a fixed charge of \$25 for operating expenses, which is the same whatever the number of workmen. Represent graphically the daily profits corresponding to the number of workmen varying from 0 to 100. From the graph, find the smallest number of workmen for which there is a net daily profit.

4. A salesman receives \$2 daily plus 5% commission on the amount of his sales. Represent his total pay corresponding to daily sales varying from 0 to \$100.

5. Some telephone companies make a flat rate of \$3 a quarter for the use of a telephone for private purposes, and charge 5¢ a message for a public telephone. Represent graphically the difference between the company's receipts per quarter for a private telephone and those of a public one, according as the number of messages on the latter varies from 50 to 200 per quarter. Show that this graph represents the change in value of $.05x - 3$ for $x = 50$ to 200.

250. Graphs of Expressions of the Form $ax + b$. When any particular expression of the form $ax + b$ is considered, it has different values according to the different values assigned to x. The corresponding values may be arranged in a table, and represented in a graph.

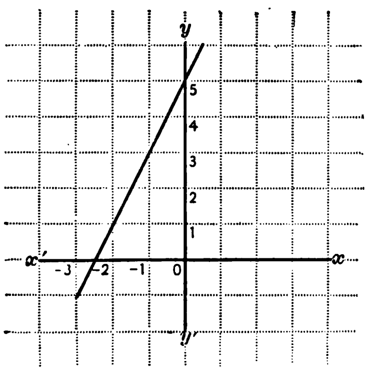


Thus, the graph at the right represents the following table of values for $3x - 2$:

TABLE

Value of x	Value of $3x - 2$
1	1
0	-2
-1	-5

It is customary to use a single letter, usually y, to represent the binomial in such problems.



Thus, if we consider $2x + 5$, we let y represent it, and speak of values of y instead of values of $2x + 5$.

The graph at the left represents the following table of values for $y = 2x + 5$:

TABLE

x	y
-3	-1
-2	1
-1	3
0	5
1	7

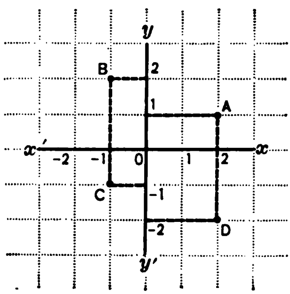
WRITTEN EXERCISES

Construct the graphs of:

1. $y = 2x + 1$.	4. $y = x + 4$.	7. $y = x$.
2. $y = 1 - 2x$.	5. $y = 5x$.	8. $y = \frac{1}{2}x + 6$.
3. $y = x - 4$.	6. $y = -3x$.	9. $y = .3x - 2$.

251. PREPARATORY.

1. In the diagram how many spaces is point A to the right of line yy'? (The symbol y' is read "y prime.") How many spaces is point A above the line xx'?



The position of point A is fixed by the distances 2, 1. (It is customary to name the distance along the line xx' first.)

2. How many spaces is point B to the left of line yy'? How many spaces above xx'?

The position of point B is fixed by the distances -1, 2.

3. How far is point C to the left of yy'? How far below xx'?

The position of point C is fixed by the distances -1, -1.

4. How far is point D from yy'? From xx'? What distances fix the position of this point?

252. Axes. The lines of reference designated by xx' and yy' are called the **axes**.

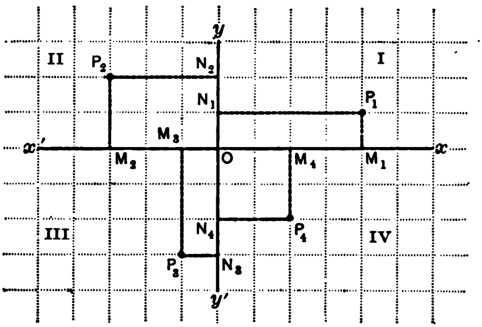
253. Quadrants. The axes divide the diagram into four quarters called *quadrants*. These are numbered I, II, III, and IV as shown in the figure on p. 211.

254. The position of the point P_1 (read P-one) is fixed by its perpendicular distances, P_1M_1, P_1N_1, from the axes. These two perpendicular distances are called the **coördinates** of point P_1. Similarly, the coördinates of P_2 are P_2M_2, P_2N_2.

The position of any point is fixed by its two coördinates.

255. The point of intersection of the axes is called the **origin of coördinates.**

256. Abscissas and Ordinates. The distance of a point from the axis yy', as P_1M_1, is called the **abscissa** of the point, and the distance of the point from the axis xx' is called the **ordinate** of the point.



The ordinates of all points in quadrants I and II are positive, and of those in III and IV are negative.

The abscissas of all points in quadrants I and IV are positive, and those of all points in II and III are negative.

257. Variables. A number symbol that may assume different values is called a **variable.**

Thus, the number of degrees of temperature from time to time, the number of hours from sunrise to sunset, the price of wheat, the number of inhabitants of the United States, are variable numbers. They measure physical or other quantities that are in themselves variable.

In mathematics, a number is variable whenever the conditions of the problem leave it free to vary.

258. Constants. A number that has a fixed value is called a **constant.**

Thus, 3, $\sqrt{5}$, -4, are constants.

259. When numbers are indicated by letters, the conditions of the problem must specify which are constant and which are variable.

It is customary to use the earlier letters of the alphabet to denote constants, and the later ones to denote variables, but it is not necessary to do so.

260. Function. When the value of one variable depends upon that of another the first variable is said to be a **function** of the second.

The function is called the **dependent** variable, and the other the **independent variable.**

For example:

1. The cost of a railroad ticket depends upon the number of miles to be traveled. That is, the cost is a function of the distance. The distance is the independent variable, and the cost is the dependent variable. Likewise, the distance one can ride depends upon the cost of the ticket, that is, the distance is a function of the cost. In the latter way of looking at it, the cost is the independent variable and the distance the dependent variable.

2. If a train moves uniformly, the distance traversed is a function of the time.

3. An iron bar expands when heated. Its length is a function of the temperature.

4. $3x - 2$ is a function of x because its value depends upon that of x.

261. "Function of x" is often briefly expressed by $f(x)$.

Thus, if

$$\begin{aligned} f(x) &= 5x - 4, \\ f(2) &= 5 \cdot 2 - 4 = 6, \\ f(0) &= 5 \cdot 0 - 4 = -4, \\ f(r) &= 5r - 4. \end{aligned}$$

262. Graphs of Functions. The **graph of a function** is a diagram representing the variation of the function due to the variation in the value of its variable.

Thus, the first graph in Sec. 250, p. 209, shows that the value of $f(x) = 3x - 2$ varies from -5 to 1 as x varies from -1 to $+1$.

263. Since the graph of the function $ax + b$ is a straight line (Sec. 215), this function is called a **linear function,** and the corresponding equation, $y = ax + b$, a linear equation.

WRITTEN EXERCISES

1. In the formula $i = prt$, if p and r are given, what quantity must change in order that i may change? Hence i is a

function of t. Construct the graph of this function when $p = \$25$ and $r = 4\%$.

2. According to Exercise 1, if p and t are given, of what is i a function? Construct the graph of this function when $p = 10$ and $t = 5$.

3. A certain fraternal insurance company charges a membership fee of \$10 and a premium of \$2 a month; hence the total cost of insurance to the policy holder is a function of the number of months. That is, cost = 2 (number of months) + 10, or $c = 2t + 10$. Construct a graph of this function of t.

4. A freight train travels 15 mi. per hour; the total distance traveled from any starting point is a function of the time, namely, $d = 15t$. Construct a graph of this function.

264. Graphs of Incomplete Equations. Either x or y may be lacking in an equation to be plotted.

Thus, the equation $y = 3$ means $y = 0x + 3$, in which $y = 3$ for all values of x; consequently the graph of $y = 3$ is a line parallel to the x-axis and 3 units above it. It is the line every point of which has the ordinate 3, just as the graph of $y = 2x$ is the line every point of which has its ordinate twice its abscissa.

Similarly, $x = 2$ means a line for which $x = 2$ irrespective of the value of y. This is a line parallel to the y-axis, and 2 units to the right of it.

Similarly, $y = -3$ means a line every point of which has the ordinate -3; it is a line parallel to the x-axis, and 3 units below it.

WRITTEN EXERCISES

Construct the graphs of:

1. $x + y = 1$.
2. $x + y = 0$.
3. $2y - 4x = 2$.
4. $x - y = 2$.
5. $x = 4y - 1$.
6. $3x + 2y = 6$.
7. $0 = x + 3$.
8. $2x - 2y = 4$.
9. $3x + 3y = 9$.

10. What line must the graph cross when $x = 0$? When $y = 0$? Construct the graph of $y = 2x - 3$ by using the points for which $x = 0$, and $y = 0$, respectively.

SUMMARY

I. Definitions and Properties.

1. *The degree of an equation* is the highest degree to which the unknowns occur in any term in the equation.

2. *A linear equation* is an equation of the first degree.

3. *A quadratic equation* is an equation of the second degree.

4. *A higher equation* is one of the third or higher degree.

5. *Absolute terms* of an equation are those which do not involve the unknowns. Sec. 239.

6. *Equivalent equations* are equations having the same roots. Sec. 240.

7. *A formula* is an expression which shows how a desired number is to be obtained from given numbers. Sec. 242.

8. $ax + b$ is the general form for all polynomials of the first degree in x. Sec. 244.

9. $ax + b = 0$ is the general equation of the first degree; and $\frac{-b}{a}$ is the general form for the root of the linear equation. Secs. 245–247.

10. In graphical work the *axes* are the lines of reference. Sec. 252.

11. *The quadrants* are the four parts into which the axes divide the plane. Sec. 253.

12. *The coördinates of a point* are its distances from the axes. Sec. 254.

13. *The origin of coördinates* is the point of intersection of the axes.

14. *The abscissa* of a point is its distance from the axis yy'.

15. *The ordinate* of a point is its distance from the axis xx'. Sec. 256.

16. *A variable* is a number symbol that may assume different values. Sec. 257.

17. A *constant* is a number that has a fixed value. Sec. 258.

18. When the value of one variable depends upon that of another the first variable is said to be a *function* of the second. The function is called the *dependent* variable and the other the *independent* variable. Sec. 260.

19. The graph of a function is a diagram representing the variation of the function due to the variation in the value of its variable. Sec. 262.

II. Processes.

1. The balance of an equation is not disturbed if a term is removed from one member and inserted with its sign changed in the other. This is called *transposing*. Sec. 236.

2. *Clearing of fractions* is freeing an equation of fractions by multiplying both members by the l. c. m. of the given denominators. Sec. 237.

3. *To solve a linear equation with one unknown:*

(1) Clear it of fractions if there are any.

(2) Remove the parentheses if there are any.

(3) Transpose the terms containing the unknowns to one member and all of the other terms to the other member.

(4) Unite the terms in each member as much as possible.

(5) Divide both members by the coefficient of the unknown.
Sec. 241.

REVIEW

WRITTEN EXERCISES

Solve for x:

1. $\frac{x+1}{x+2}+\frac{x+3}{x+4}=2.$

2. $(a-x)(b+x)=b-x^2.$

3. $(a+x)(b+x)=ab+x^2.$

4. $(a-x)(b-x)=b+x^2.$

5. $(a+bx)(a-bx)=(a^2+b^2x)(a-x).$

6. $(a+bx)(b-ax)=(a-bx)(ax-b).$

7. $(1-x)(2-x)-(3-x)(4-x)=5-x.$

8. $(a+x)(b-x)=a-x^2$.

9. $ax(bx+1)=bx(ax+1)+a+b$.

10. $(1+x)(2+x)+(5+6x)(1+8x)=(4+7x)^2$.

11. $(a+x)(c+d+x)-(a-x)(c-d-x)=ax+cx+dx$.

12. How much water must be added to 80 pounds of a 5 per cent solution of salt to obtain a 4 per cent solution?

13. Two men are 25 miles apart and walk toward each other at the rates of $3\frac{1}{2}$ and 4 miles an hour respectively. After how long do they meet?

14. A man travels 50 miles in an automobile in $3\frac{1}{4}$ hours. If he runs at the rate of 20 miles an hour in the country, and at the rate of 8 miles an hour when within city limits, find how many miles of his journey is in the country.

15. In a 50-mile race between two yachts their rates are as 5 to 4. One yacht has a start of 20 minutes, but is beaten by 4 miles. Find the rate of each.

16. A train running 30 miles an hour requires 21 minutes longer to go a certain distance than does a train running 36 miles per hour. What is this distance?

17. What is that number which when divided by 3 is equal to one quarter of the sum of itself and 24?

18. A sum of $1050 is divided into two parts and invested; the simple interest on the one part at 4% for 6 years is the same as the simple interest on the other at 5% for 12 years. Find how the money is divided.

19. A and B set out on an automobile trip, A having $\frac{2}{3}$ as much money with him as B; after A had paid out $1 less than $\frac{2}{3}$ of his money, and B had paid $1 more than $\frac{7}{8}$ of his, it was found that B had left only half as much as A. How much had each at the outset?

20. In forming a regiment into a solid square 60 were left over; but, when formed into a rectangle with 5 men more in front than before and 3 less in depth, there was one man wanting to complete it. Find the number of soldiers in the regiment.

21. A man is to pay $10,000 Jan. 1. He wishes to prepay this in two equal amounts on April 1 and Aug. 1 next preceding. Interest on the payment being allowed at 5% per annum for the period paid in advance, what should each payment be?

22. Two automobiles start from the same place; one goes east at the rate of 18 mi. an hour and the other west at 15 mi. an hour. In how many hours are they 330 mi. apart?

23. A train 660 ft. long running at a speed of 15 mi. an hour will pass completely through the Simplon tunnel in Switzerland in $49\frac{1}{2}$ minutes. How long is the tunnel?

24. A train going from New York to Chicago at the average rate of 40 mi. an hour takes $4\frac{4}{5}$ hr. longer than one going 50 mi. an hour. Find the distance between these places.

25. Given $a =$ the amount, $r =$ the rate, and $p =$ the principal in a problem of simple interest, what is the time?

26. In Exercise 25 let $a = \$560$, $r = 4\%$, and $t = 3$ yr. Find the principal.

27. Divide the number a into two parts such that m times the greater may exceed n times the less by b.

28. A quantity of metal composed of copper and tin weighs 80 lb.; for every 7 lb. of copper there are 3 lb. of tin. How many pounds of copper must be added to the mass to make a metal containing 11 lb. of copper to 4 lb. of tin.

29. To a mass of metal composed of 4 parts of silver to 1 part of tin enough silver is added to make a mass containing 6 parts of silver to 1 part of tin. How many ounces of silver are added per ounce of the original metal?

SUPPLEMENTARY WORK

ADDITIONAL EXERCISES

1. A certain number consists of two digits whose difference is 3; and, if the digits are interchanged, the number so formed is $\frac{4}{7}$ of the original number. Find the latter.

SOLUTION. 1. Let x be the smaller digit.

2. Then $3 + x$ is the greater.
3. $\therefore 10(x + 3) + x$ is the given number.
4. $\therefore 10x + (x + 3)$ is the number with the digits interchanged.
5. According to the problem, $10x + (x + 3) = \frac{4}{7}[10(x + 3) + x]$.
6. $\therefore$ performing the operations, $77x + 21 = 44x + 120$.
7. Solving (6), $33x = 99$, and $x = 3$.
8. $\therefore$ $x + 3 = 6$, and the number is 63.

TEST. The number, 63, satisfies the conditions of the problem, for $6 - 3 = 3$, and $36 = \frac{4}{7}$ of 63.

2. The units' digit of a certain number exceeds the tens' digit by 4, and when the number is divided by the sum of its digits, the quotient is 7. Find the number.

3. There is a number of two digits in which the tens' digit exceeds the units' digit by 2, and if the number is diminished by $\frac{3}{2}$ of the sum of its digits, they are interchanged. Find the number.

4. Separate 150 into two numbers such that if one be divided by 23 and the other by 27, the sum of the quotients is 6.

5. The sum of three numbers, a, b, c, is 3036; a is the same multiple of 7 that b is of 4, and also the same multiple of 5 that c is of 2. What are the numbers?

SUGGESTION. Show that a, b, and c may be represented by $7x$, $4x$, and $\frac{7x}{5} \cdot 2$.

6. A number is composed of three digits each greater by one than that on its right; the difference between the number and $\frac{1}{4}$ of the number formed by reversing the order of the digits is 36 times the sum of the digits. Find the number.

SUGGESTION. A convenient selection of the digits is $x + 1$, x, $x - 1$.

7. Find three consecutive integers whose sum is 9 greater than twice the largest integer.

8. At what time between 3 and 4 o'clock are the hands of a clock together?

Solution. 1. The minute hand moves 1 minute space per minute.

2. The hour hand moves $\frac{1}{12}$ of a minute space per minute.

3. Let $x =$ the number of minutes after 3 o'clock when the hands are together.

4. x is the number of spaces moved by the minute hand.

5. $\frac{x}{12}$ is the number of spaces moved by the hour hand.

6. The hour hand is now $15 + \frac{x}{12}$ minute spaces from XII and the minute hand is x spaces from XII.

7. But they are to be together, hence $x = 15 + \frac{x}{12}$.

8. $\therefore x = 16\frac{4}{11}$, and the hands are together $16\frac{4}{11}$ minutes past 3 o'clock.

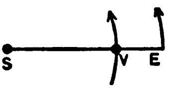
9. How many minute spaces must the minute hand gain on the hour hand from the time they meet until they lie opposite to each other in the same straight line? At what time are the hands opposite each other for the first time after 12 o'clock?

10. At what time between 7 and 8 o'clock are the hands of a clock opposite each other?

11. At how many different times, and when, are the hands of a clock at right angles between 4 and 5 o'clock?

12. A and B enter a race together; at the end of 5 minutes A is 900 yd. from the starting line and 75 yd. ahead of B; at this point he falls, and though he renews the race, his rate is 20 yd. a minute less for the rest of the course; he crosses the line $\frac{1}{2}$ minute after B. How long did the race last?

13. In astronomy it is important to know when planets are in line between the earth and the sun. This is called *conjunction.* Taking the earth's time of revolution about the sun as 365 days and that of Venus as 225 days, how long after one conjunction of Venus until the next one occurs?



Suggestion. The problem is quite analogous to that of the hands of a watch. For the purposes of this problem we suppose all the planets to move in the same plane and

in circular paths (orbits) in the same direction of revolution about the sun as a center.

1. Let $x =$ the number of days.

2. Venus will have made $\frac{x}{225}$ revolutions.

3. The earth will have made $\frac{x}{365}$ revolutions.

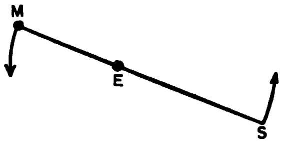
4. But to be in conjunction, Venus (which goes faster) must have gone as far as the earth and 1 revolution besides.

$$\text{Hence, } \frac{x}{225} = \frac{x}{365} + 1,$$

$$\text{and} \quad x = 586\tfrac{1}{2}\tfrac{7}{8}.$$

14. Taking 88 days as Mercury's time of revolution about the sun, how long from one conjunction of Mercury to the next?

15. When the earth is between a planet and the sun, in the same line with them, the planet is said to be in opposition to the sun. Taking 687 days as Mars' time of revolution about the sun, how long is it from one opposition of Mars to the next?



16. Taking 4307 days as Jupiter's time of revolution about the sun, how long is it from one opposition of Jupiter to the next?

17. Answer the same question for Saturn, whose time of revolution is 28.5 yr.

18. Also for Neptune, whose time of revolution is 165.5 yr.

19. Pythagoras being asked the time of day, replied: "There remains of the day (from 6 A.M. to 6 P.M.) twice the number of hours already passed." What time was it?

20. The three Graces, carrying 4 apples each, met the 9 Muses; they gave each Muse the same number of apples; then the Muses and Graces had equal shares. How many had each?

21. A fruit vender gave a boy 4 dozen oranges to sell and agreed to pay him $\frac{3}{4}$¢ for each orange sold, but demanded a payment of 2¢ for each orange eaten; the boy disposed of all the oranges and received 25¢. How many oranges did he eat?

22. A robber in escaping from a castle met a guard whom he bribed with $\frac{1}{5}$ of his plunder; at the next gate he bribed another guard with $\frac{1}{4}$ of the plunder remaining; at the third gate he bribed another guard with $\frac{1}{3}$ of the plunder remaining; the robber then escaped with 2000 ducats. How many ducats did he steal?

23. A servant agreed to work for £10 a year and his livery. At the end of 7 months his lord discharged him, giving him the livery only. How many pounds was the livery worth?

24. Alcides was asked, "How many are there of your numerous herd?" He replied: "If I had 6 less than twice as many more, the number would be 306." How many were there in his herd?

25. A courier went from Paris to Grenoble, 120 leagues, in 4 days, each day's journey being 2 leagues shorter than that of the preceding day. How many leagues did he travel each day? (Ozanam's *Algebra*, 1702.)

26. A workman can perform a units of work in b units of time; a second workman can perform c units of work in d units of time; a third can do e units of work in f units of time. In what time will these three workmen together perform g units of work? (Clairaut's *Algebra*, 1746.)

27. Two messengers, A and B, set out towards each other from two places 59 mi. apart, B starting 1 hr. after A. A goes 7 mi. in 2 hr., and B 8 mi. in 3 hr. How far will A have gone when he meets B? (Newton's *Arithmetica Universalis*, 1707.)

CHAPTER XV

SYSTEMS OF EQUATIONS

EQUATIONS WITH TWO UNKNOWNS

265. Solving General Equations. We have solved particular equations with two unknowns (Chapter X). The following treatment leads to *the solution of the general equations with two unknowns.*

266. General Form of Linear Equations with Two Unknowns. A *general form* for an equation of the first degree with two unknowns is

$$ax + by = e.$$

A *general form* for two such equations is

$$\begin{cases} ax + by = e, & (1) \\ cx + dy = f. & (2) \end{cases}$$

That is, by filling the blanks denoted by a, b, c, d, e, f (see Sec. 243), we obtain a set of two particular equations of this kind.

267. Solution by Formula. From the general equations (*1*) and (*2*) it is possible (without knowing the values of a, b, c, d, e, f) to find a *general form* for the solution.

Solve:

$$\begin{cases} ax + by = e, & (1) \\ cx + dy = f. & (2) \end{cases}$$

Multiplying (*1*) by c, and (*2*) by a,

$$cax + cby = ce, \qquad (3)$$
$$acx + ady = af. \qquad (4)$$

Subtracting (*3*) from (*4*),

$$ady - cby = af - ce. \qquad (5)$$

Thus,

$$(ad - bc)y = af - ce. \qquad (6)$$

Dividing by the coefficient of y,

$$y = \frac{af - ce}{ad - bc}. \qquad (7)$$

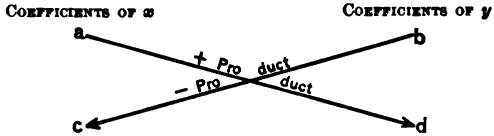
Eliminating by multiplying (*1*) by d, and (*2*) by b, and subtracting,

$$x = \frac{de - bf}{ad - bc}. \qquad (8)$$

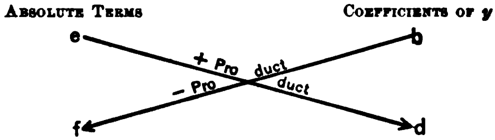
These results are the *formulas for the roots* of any system of two linear simultaneous equations with two unknowns.

268. The application of these formulas is made easier by noticing how they are formed from the known numbers in the equations.

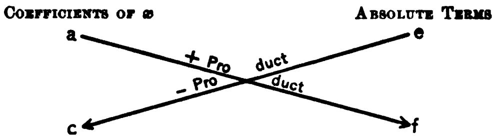
1. The denominator is the same in each result and is made up from the coefficients as follows:



2. The numerator of the value of x is made up thus:



3. The numerator of the value of y:



The following examples will illustrate the use of the formulas in solving equations:

1. Solve: $\begin{cases} 2x - 3y = 4, & (1) \\ 4x + 2y = 1. & (2) \end{cases}$

$$x = \frac{\begin{array}{cc} 4 & -3 \\ 1 & +2 \end{array}}{\begin{array}{cc} 2 & -3 \\ 4 & 2 \end{array}} = \frac{4 \cdot 2 - 1(-3)}{2 \cdot 2 - (-3) \cdot 4} = \frac{8 + 3}{4 + 12} = \frac{11}{16}. \qquad (3)$$

$$y = \frac{\begin{array}{cc} 2 & 4 \\ 4 & 1 \end{array}}{\begin{array}{cc} 2 & -3 \\ 4 & 2 \end{array}} = \frac{2 \cdot 1 - 4 \cdot 4}{2 \cdot 2 - 4\,(-3)} = \frac{2 - 16}{16} = \frac{-14}{16} = \frac{-7}{8}. \qquad (4)$$

TEST. $\frac{2 \cdot 11}{16} - \frac{3 \cdot -7}{8} = \frac{64}{16} = 4$; $\frac{4 \cdot 11}{16} + \frac{2 \cdot -7}{8} = \frac{16}{16} = 1.$

2. Solve: $\begin{cases} 4x + 7y = -27. & (1) \\ x - 2y = 12. & (2) \end{cases}$

$$x = \frac{\begin{vmatrix} -27 & 7 \\ 12 & -2 \end{vmatrix}}{\begin{vmatrix} 4 & 7 \\ 1 & -2 \end{vmatrix}} = \frac{-27 \cdot (-2) - 7 \cdot 12}{4 \cdot (-2) - 7} = \frac{-30}{-15} = 2. \qquad (3)$$

$$y = \frac{\begin{vmatrix} 4 & -27 \\ 1 & 12 \end{vmatrix}}{-15} = \frac{4 \cdot 12 - 1(-27)}{-15} = \frac{75}{-15} = -5. \qquad (4)$$

TEST. $4 \cdot 2 + 7(-5) = -27$; $2 - 2(-5) = 12$.

269. This method usually gives the values by inspection, for the products and differences can be read off from the equations themselves.

For example:

$4x + 2y = 1,$ $3x - 5y = 4.$	$4x + 2y = 1,$ $3x - 5y = 4.$	$4x + 2y = 1,$ $3x - 5y = 4.$
Denominator $= 4(-5) - 2 \cdot 3$ $= -26.$	Numerator of y $= 4 \cdot 4 - 1 \cdot 3 = 13.$ $\therefore y = \frac{13}{-26} = -\frac{1}{2}.$	Numerator of x $= 1(-5) - 2 \cdot 4 = -13$ $\therefore x = \frac{-13}{-26} = \frac{1}{2}.$

WRITTEN EXERCISES

Solve:

1. $2x + 7y = 11,$
   $5x - 9y = 2.$
2. $3x + 7y = -1,$
   $2x - 3y = 4.$
3. $4x - 7y = 5,$
   $8x + 15y = 16.$
4. $2x - y = 6,$
   $10x - 3y = 3.$
5. $7x + 9y = 15,$
   $8x - 4y = 12.$
6. $2x - 6y = 1,$
   $4x + 2y = 8.$
7. $2x + y = 1,$
   $x + 2y = 2.$
8. $11x + 22y = 33,$
   $4x + 18y = 24.$
9. $1.4x + 2.1y = 1,$
   $2.8x + 3.3y = 2.$
10. $30x + 25y = 18,$
    $13x + 16y = 45.$
11. $9x - 12y = 16,$
    $21x - 35y = 24.$
12. $3x + 5y = 184,$
    $10x - 5y = -15.$

13. $12x+6y=8,$
$48x-9y=92.$

14. $x+17y=300,$
$11x-y=104.$

15. $3x-5y=405,$
$6x+2y=195.$

16. $12x+6y=8,$
$48x-9y=29.$

17. $5x+45y=-6,$
$15x-9y=1.$

18. $\frac{x}{a}+\frac{y}{b}=1,$

$\frac{x}{2a}-\frac{y}{3b}=4.$

19. $2x-3y=5a-b,$
$3x-2y=b+5a.$

20. $26x+42y=33,$
$39x+28y=44.$

21. $3p+q=3,$
$5p-q=13.$

22. $4h-2b=1,$
$3h+b=5.$

23. $4v-5w=6,$
$-3v+4w=1.$

24. $14R+3r=2,$
$9R+2r=6.$

25. $(a+c)x-(a-c)y=2ab,$
$(a+b)x-(a-b)y=2ac.$

26. An investor purchases two kinds of securities; one kind pays 2% and the other 4% annually; his annual income from both sources is \$900; if he had invested as much in the 2% securities as he did in the 4 per cents, and *vice versa*, his income would have been \$600. How many dollars were invested in each kind?

Solution. 1. Let x and y be the number of dollars invested at 2% and 4% respectively.

2. Then, by the conditions of the problem, $.02x+.04y=900,$
3. and $.04x+.02y=600.$
4. Multiplying (2) by 2, $.04x+.08y=1800.$
5. Subtracting (3) from (4), $.06y=1200.$
6. Solving (5), $y=20{,}000.$
7. Substituting $y=20{,}000$ in (1), $.02x+800=900.$
8. Solving (7), $x=5000.$

$\therefore$ he invested \$5000 at 2% and \$20,000 at 4%.

Test. 2% of \$5000 + 4% of \$20,000 = \$900;
2% of \$20,000 + 4% of \$5000 = \$600.

27. An investor purchased Pennsylvania Railroad stock which pays an annual dividend of 6%, and Metropolitan Railway bonds paying 4%; his annual income from both was \$2100; if the stock had paid 1% less and the bonds 1% more, his total income would have been \$2000. How much did he invest in each?

28. An investment of \$20,000 in one stock, and one of \$10,000 in another together yield \$1300 annually; an investment of half as much in the first and twice as much in the second would together yield \$1100 annually. What is the annual rate of dividend in each stock?

29. A man has a certain sum of money invested at 5%; he reinvests the whole sum, placing three times as much of it at 8% as he does at 4%. His income is increased \$200 a year by the change. How much money has he, and what is his income?

30. A standard daily ration for an adult laborer requires 4 oz. of protein and 4 oz. of fat.

The following table shows the approximate amounts of protein and fat in various foods:

Food	Per Cent of Fat	Per Cent of Protein
Mutton	37	14
Pork (fresh)	26	13
Eggs	9	13
Bread (white)	1	9
Beans (dried)	2	22
Corn (green)	1	3
Rice	$\frac{1}{2}$	8

How many ounces of mutton and bread are needed to make a standard ration for one day?

Solution. 1. Let x and y be the numbers of ounces required of mutton and bread respectively.

2. Then $0.14x + 0.09y$ is the amount of protein in these foods according to the table.

3. Also $0.37\,x + 0.01\,y$ is the fat in these foods, according to the table.

4. Thus, $0.14\,x + 0.09\,y = 4,$

5. and $0.37\,x + 0.01\,y = 4.$

6. Multiplying (4) by 100, $14\,x + 9\,y = 400.$

7. Multiplying (5) by 100, $37\,x + y = 400.$

8. Subtracting (7) from (6), $23\,x - 8\,y = 0.$

9. $\therefore x = \frac{8\,y}{23}.$

10. Using (9) in (7), $\frac{37 \cdot 8\,y}{23} + y = 400.$

11. From (10), $y = 28.9$ (to nearest 0.1).

12. From (11) and (9), $x = 10$ (to nearest 0.1)

13. $\therefore$ the ration is 10 oz. of mutton and 28.9 oz. of bread.

A negative result for either unknown quantity would show that it is impossible to make up the standard ration out of the foods named.

Find which of the following combinations of foods can make a standard ration, and the number of ounces of each food required:

31. Mutton and beans.

32. Mutton and rice.

33. Bread and eggs.

34. Pork and beans.

35. Eggs and corn.

36. Mutton and corn.

37. Bread and pork.

38. Bread and rice.

39. Pork and rice.

40. Eggs and rice.

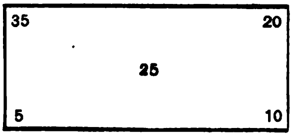
41. In what proportion should cream containing 35 % of fat, and milk containing 5 %, be mixed to produce cream containing 25 % ?

It is a common practice to "standardize" cream before selling; that is, to make the fat content a certain desired percentage, by mixing two qualities of cream or milk, one richer and one poorer than the desired standard quality.

The following rule has been given for the solution of problems like the preceding:

Draw a rectangle and write at the two left-hand corners the percentage of fat in the two fluids to be mixed ; and in the center place the percentage desired. At each of the other two corners write the positive

differences of the two numbers standing in the same diagonal with that corner. The numbers thus found for the right-hand corners show the relative amounts that should be used of the fluid whose percentage stands at the corresponding left corner.



The accompanying diagram indicates that in the preceding problem 20 parts of cream should be used to 10 parts of milk.

42. Prove that the rule given is correct.

43. In what proportion should cream containing 35 % of fat and cream containing 20 % of fat be mixed to produce cream containing 25 %?

270. Solving Fractional Equations. I. *In case of fractional equations, when the unknown quantities occur only in monomial denominators, it is best not to clear of fractions.*

EXAMPLES

1. Solve:

$$\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=2. & (1) \\ \dfrac{1}{2x}-\dfrac{1}{3y}=\dfrac{5}{36}. & (2) \end{cases}$$

Multiplying (2) by 4, $$\frac{4}{2x}-\frac{4}{3y}=\frac{20}{36}. \quad (3)$$

Simplifying (3), $$\frac{2}{x}-\frac{4}{3y}=\frac{5}{9}. \quad (4)$$

Subtracting (4) from (1), $$\frac{13}{3y}=\frac{13}{9}. \quad (5)$$

Dividing (5) by 13, $$\frac{1}{3y}=\frac{1}{9}. \quad (6)$$

Clearing (6) of fractions, $$3y=9. \quad (7)$$

$$\therefore y=3. \quad (8)$$

Substituting 3 for y in (1), $$x=2. \quad (9)$$

TEST. $$\frac{2}{2}+\frac{3}{3}=2 \text{ and } \frac{1}{2\cdot 2}-\frac{1}{3\cdot 3}=\frac{5}{36}.$$

It is evident that the above equations, if cleared of fractions, would contain terms in xy which would complicate the solution.

2. In such equations the unknown quantities may be thought of as $\frac{1}{x}$ and $\frac{1}{y}$.

Thus, to solve $\frac{10}{x}+\frac{9}{y}=5$, and $\frac{35}{x}-\frac{6}{y}=5$, is to solve

$$10\left(\frac{1}{x}\right)+9\left(\frac{1}{y}\right)=5, \text{ and } 35\left(\frac{1}{x}\right)-6\left(\frac{1}{y}\right)=5.$$

The coefficients are now manipulated as in the case of integral equations. It is often convenient to introduce new unknowns in such problems; in this case, for example, by putting

$$\frac{1}{x}=x' \text{ and } \frac{1}{y}=y',$$

the equations become $\quad 10\,x'+9\,y'=5,$

and $\quad 35\,x'-6\,y'=5.$

3. Solve:

$$\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=m, & (1) \\[2ex] \dfrac{c}{x}+\dfrac{d}{y}=n. & (2) \end{cases}$$

Regarding $\frac{1}{x}$ and $\frac{1}{y}$ as unknowns, and using the method of Sec. 268,

$$\frac{1}{x}=\frac{\begin{matrix} m & b \\ n & d \end{matrix}}{\begin{matrix} a & b \\ c & d \end{matrix}}=\frac{md-bn}{ad-bc}. \qquad (3)$$

$$\frac{1}{y}=\frac{\begin{matrix} a & m \\ c & n \end{matrix}}{\begin{matrix} a & b \\ c & d \end{matrix}}=\frac{an-mc}{ad-bc}. \qquad (4)$$

From (3), $$x=\frac{ad-bc}{md-bn}. \qquad (5)$$

From (4), $$y=\frac{ad-bc}{an-mc}. \qquad (6)$$

Test. In literal equations it is generally more convenient to test by reworking (when possible, by a different method or in a different order) than by substituting the values.

WRITTEN EXERCISES

Solve:

1. $\frac{7x}{5}+\frac{5y}{6}=2,$

   $\frac{3x}{10}-\frac{5y}{12}=4.$

2. $\frac{1}{x}+\frac{1}{y}=\frac{3}{4},$

   $\frac{1}{x}-\frac{1}{y}=\frac{1}{4}.$

3. $\frac{6}{x}+\frac{8}{y}=3,$

   $\frac{12}{x}-\frac{20}{y}=-3.$

4. $\frac{1}{3x}+\frac{1}{2y}=-\frac{1}{6},$

   $\frac{1}{x}+\frac{2}{y}=-1.$

5. $\frac{1}{x}+\frac{a}{y}=2,$

   $\frac{b}{x}-\frac{1}{y}=a.$

6. $\frac{2a}{x}-\frac{3b}{y}=4,$

   $\frac{3a}{x}-\frac{4b}{y}=2c.$

7. $\frac{3}{x}-\frac{2a}{y}=b,$

   $\frac{5}{2x}+\frac{3b}{y}=c.$

8. $2x+\frac{1}{y}=14,$

   $4x-\frac{5}{y}=14.$

   SUGGESTION. Regard x and $\frac{1}{y}$ as the unknowns.

9. $\frac{6}{x}-2y=6,$

   $\frac{9}{x}+y=1.$

10. $\frac{5}{x}+3y=17,$

    $\frac{2}{x}-2y=-6.$

11. $\frac{3}{x}+\frac{2}{y}=-12,$

    $\frac{4}{x}-\frac{3}{y}=1.$

12. $\frac{1}{x}+\frac{a}{y}=1,$

    $\frac{a}{x}+\frac{1}{y}=1.$

13. The sum of the reciprocals of two numbers is $\frac{5}{4}$, and the difference of the reciprocals is $\frac{1}{4}$. Find the numbers.

14. Find two numbers such that 3 times the reciprocal of the first added to 5 times the reciprocal of the second makes 2, and 24 times the reciprocal of the first diminished by 10 times the reciprocal of the second makes 1.

271. II. *It is usually best to clear of fractions when the unknowns occur in polynomial denominators.*

EXAMPLES

1. Solve: $\begin{cases} \dfrac{2x+1}{x-4}=\dfrac{2y+18}{y+6}, & (1) \\ \dfrac{x+1}{y}=-2. & (2) \end{cases}$

Clearing (1) of fractions, $2xy+y+12x+6=2xy+18x-8y-72.$ (3)

Collecting terms in (3), $-6x+9y=-78.$ (4)

From (2), $x+2y=-1.$ (5)

Solving (4) and (5), $x=7,\ y=-4.$ (6)

2. Solve: $\begin{cases} \dfrac{2x+6}{x}=y+\dfrac{10x+6}{x}. & (1) \\ \dfrac{x+4}{y+3}=-2. & (2) \end{cases}$

Separating the fractions of (1), $\dfrac{2x}{x}+\dfrac{6}{x}=y+\dfrac{10x}{x}+\dfrac{6}{x}.$ (3)

Simplifying the terms of (3), $2+\dfrac{6}{x}=y+10+\dfrac{6}{x}.$ (4)

Hence, $2=y+10.$ (5)

Or, $y=-8.$ (6)

Substituting $y=-8$ in (2), and solving, $x=6.$ (7)

WRITTEN EXERCISES

Solve:

1. $\dfrac{2x+3y}{1+4y}=1,$ $\dfrac{4x-2y}{x-y}=-2.$

2. $\dfrac{x+1}{y}=\dfrac{x+5}{y+2},$ $\dfrac{2x-3y}{4x-2y}=\dfrac{1}{2}.$

3. $\dfrac{x+1}{y}=4,$ $\dfrac{y+1}{x}=3.$

4. $\dfrac{x}{y}=\dfrac{5}{4},$ $\dfrac{x+1}{y+1}=\dfrac{16}{13}.$

5. $\dfrac{x+5}{y-3}=\dfrac{x+1}{y-2},$ $\dfrac{x+2}{y-3}=3.$

6. $\dfrac{x+1}{5y}=-1,$ $\dfrac{3x+2y}{y+6}=2.$

7. $y+1+\dfrac{1}{3x}=2y-\dfrac{27x-3}{9x},$ $\dfrac{1}{3}+\dfrac{y}{2x}=\dfrac{x}{3}-\dfrac{30x-5y}{10x}.$

8. $\frac{x+2y}{2y-x}=\frac{1}{7},\quad y+\frac{x-3y}{3y}+\frac{1}{2}=0.$

9. $\frac{c-2y}{a+b-2x}\cdot\frac{c}{b-a}=-1,\quad \frac{c-2y}{b-2x}\cdot\frac{c}{b}=-1.$

10. $\frac{y}{x+a}\cdot\frac{c}{a}=1,\quad \frac{y}{x-a}=-\frac{c}{a}.$

What are the values of x and y thus found when $a=3$ and $c=5$?

11. $\frac{2x}{3}-\frac{5y}{12}-\left(\frac{3x}{2}-\frac{4y}{3}\right)=-\frac{2}{3},\quad \frac{x-y}{x+y}=\frac{1}{5}.$

12. $3x+4ay=8ab,\quad \frac{x}{7a}-7y+3b=0.$

13. $\frac{2}{x}+\frac{1}{y}=2,\quad \frac{3}{x}-\frac{2}{y}=5.$

14. $\frac{2x}{3}-4+\frac{y}{2}+x=8-\frac{3y}{4}+\frac{1}{12},\quad \frac{y}{6}-\frac{x}{2}+2=\frac{1}{6}-2x+6.$

GRAPHS OF EQUATIONS WITH TWO UNKNOWNS

272. Preparatory.

1. The diagram shows the general trend in the increase of the pig-iron product in the United States, Germany, and England during a period of ten years.

(*a*) In which country has the increase been the most rapid? The least rapid?

(*b*) In what year was the amount produced in England and the United States the same?

(*c*) Answer the same question for England and Germany.

2. A factory has a fixed charge of \$ 20 daily, and makes an average gross profit of \$ 2 daily per workman employed. What remains after the fixed charge is paid is net profit. Make a graphic representation of how the net profit varies as the number of workmen varies from 0 to 50.

SOLUTION. 1. Let $w =$ the number of workmen.

2. Then, $2w =$ the gross profit,

3. and $2w - 20 =$ the gross profit less the fixed charges, or the net profit.

4. Let $p =$ the net profit.

5. Then $p = 2w - 20$.

The graph exhibits the change in p due to a change in w subject to the above relation; or it is the graph of the equation $p = 2w - 20$.

3. A second factory has a fixed charge of \$ 60 and makes an average gross profit of \$ 3 daily per workman employed. Construct a graph to represent the net profits of the second factory in the same diagram as in Exercise 1.

What is the relation between p and w, of which this is the graph?

4. From the diagram read:

(a) The number of workmen for which each factory makes the same net profit. The amount of that profit.

(b) The number of workmen for which the first factory makes the larger net profit.

(c) The number of workmen for which the second factory makes the larger net profit.

(d) The net profit of each factory where 25 workmen are employed. 40 workmen. 15 workmen. 10 workmen. 5 workmen.

(e) The number of workmen each factory must employ to make \$ 30 net profit. Also \$ 60 net profit.

273. The process illustrated above can be applied in the graphic solution of any set of two simultaneous linear equations with two unknowns.

For example:

$$\begin{cases} x - y = 1, & (1) \\ x + 2y = 4. & (2) \end{cases}$$

We have already seen how to represent graphically all of the solutions of an equation of the first degree in two unknowns.

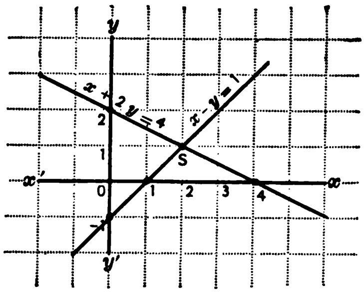
If we represent the equation $x - y = 1$, and the equation $x + 2y = 4$, in the same diagram it is possible at once to read the values of x and y that satisfy both equations, namely the values which correspond to the point s that lies on both graphs.

Table of Values for Equation (1)

x	y
0	-1
1	0
2	1

Table of Values for Equation (2)

x	y
0	2
1	$1\frac{1}{2}$
2	1



Graph

ORAL EXERCISES

1. Read from the graph of the first equation the value of y when $x = 0$. When $x = 1$. When $x = 2$. When $x = 3$. When $x = 4$.

2. How many points are needed to fix the position of a straight line?

3. Read from the graph of the second equation the value of y when $x = 0$. When $x = 2$. When $x = 4$.

4. What point in the diagram is common to the two graphs? What are the values of x and y for point s?

5. Since the values of x and y for any point on the graph of an equation satisfy the equation, what values of x and y satisfy both of the above equations?

6. What point in the diagram represents the solution of the system of equations?

274. The solution of two simultaneous equations of the first degree in two unknowns is represented graphically by the point of intersection of the graphs of the equations.

NOTE. Since the accuracy of the graphical solution depends upon the precision of the diagram, the results so found must be regarded as approximate. Their accuracy must be tested in the usual way. The graphical method of solution is of practical use chiefly in applied mathematics where an approximate result is commonly sufficient, but in theoretic mathematics it is important as exhibiting clearly to the eye the relations between the variables involved.

WRITTEN EXERCISES

Solve graphically:

1. $x+y=5,$ $3x-2y=0.$	3. $x+y=0,$ $y+3=4.$	5. $x+y=7,$ $2x-y=5.$
2. $3x-y=1,$ $x+2y=12.$	4. $x+2y=6,$ $2x-y=2.$	6. $3x+y=12,$ $x-y=0.$

EQUATIONS WITH THREE OR MORE UNKNOWNS

275. The definitions and methods given for the solution of two equations with two unknowns may be applied equally well to a greater number of equations and unknowns.

To solve three linear equations with three unknowns, eliminate one unknown from any pair of the equations and the same unknown from any other pair; two equations are thus formed which involve only two unknowns and which may be solved by methods previously given.

Four or more equations with four or more unknowns may be solved similarly.

EXAMPLE

Solve:
$$\begin{cases} x - 2y + 3z = 2, & (1) \\ 2x - 3y + z = 1, & (2) \\ 3x - y + 2z = 9. & (3) \end{cases}$$

To eliminate x from (1) and (2):

Multiplying (1) by 2, $2x - 4y + 6z = 4.$ (4)

Subtracting (2) from (4), $-y + 5z = 3.$ (5)

To eliminate x from (1) and (3):

Multiplying (1) by 3, $3x - 6y + 9z = 6.$ (6)

Subtracting (3) from (6), $-5y + 7z = -3.$ (7)

To eliminate y from (5) and (7):

Multiplying (5) by 5, $-5y + 25z = 15.$ (8)

Subtracting (7) from (8), $18z = 18.$ (9)

$\therefore z = 1.$ (10)

To eliminate z from (8):

Substituting $z = 1$ in (8), $-5y + 25 = 15.$ (11)

Solving (11), $y = 2.$ (12)

Substituting $y = 2$, $z = 1$ in (1), $x - 4 + 3 = 2.$ (13)

Solving (13), $x = 3.$ (14)

TEST. (2) $2 \cdot 3 - 3 \cdot 2 + 1 \cdot 1 = 1.$

(3) $3 \cdot 3 - 1 \cdot 2 + 2 \cdot 1 = 9.$

Since x was found in step (13) by substituting $y = 2$, $z = 1$ in equation (1), it is not necessary to substitute again in this equation when testing the results.

276. Literal equations are solved in the same way; when fractions are involved, the equations are treated as shown in Secs. 270 and 271.

WRITTEN EXERCISES

Solve:

1. $2x + 3y + 4z = 20,$
   $3x + 4y + 5z = 26,$
   $3x + 5y + 6z = 31.$

2. $x + y + z = 5,$
   $x + y - z = 7,$
   $x - y - z = 3.$

3. $x + 2y = 7,$
   $y + 2z = 2,$
   $3x + 2y = z - 1.$

4. $5x + 3y = 65,$
   $2y - z = 11,$
   $3x + 4z = 57.$

5. $y + z = -a,$
$x + z = -b,$
$x + y = -c.$

6. $x + y - z = 1,$
$8x + 3y - 6z = 1,$
$4x + y - 3z = 1.$

7. $x + y + 2z = 2(b + c),$
$x + 2y + z = 2(a + c),$
$2x + y + z = 2(a + b).$

8. $\frac{1}{2}x + \frac{1}{3}y = 12 - \frac{1}{6}z,$
$\frac{1}{2}y + \frac{1}{3}z = 8 + \frac{1}{6}x,$
$\frac{1}{2}x + \frac{1}{3}z = 10.$

9. $x + ay = b,$
$ax + z = c,$
$z + cy = a.$

10. $x + 2y = 24,$
$2x + z = 12,$
$y + z = 19.$

11. $x + \frac{1}{2}y = 100,$
$y + \frac{1}{3}z = 100,$
$z + \frac{1}{4}x = 100.$

12. $7x + 13y = 205,$
$14x + 5z = 300,$
$12y + 20z = 140.$

13. $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 31,$
$\frac{x}{3} + \frac{y}{4} + \frac{z}{5} = 23.5,$
$\frac{x}{4} + \frac{y}{5} + \frac{z}{6} = 19.$

14. $x^2 + xy + z = 2,$
$x + 2y + z = 3,$
$x - y + z = 0.$

SUGGESTION. Begin with the last two equations.

15. $x + z = 1 - cy,$
$y + z = -cx,$
$x + y = -1 - cz.$

16. $x = 6 + \frac{y}{3},$
$y = 4 + \frac{z}{5},$
$z = 8 + \frac{x}{4}.$

17. $4x + 9y + 3z = -4,$
$8x + 12y - 7z = -4,$
$12x + 15y = -4.$

18. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1,$
$\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = -3,$
$\frac{3}{x} - \frac{4}{y} - \frac{5}{z} = 14.$

SUGGESTION. First solve for $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}.$

19. $ax + by = 1,$
$cy - az = 1,$
$bz - cx = 1.$

SUGGESTION. Multiply the equations by c, b, a, respectively, and add.

20. The three chief ingredients of the standard daily ration for a laborer are 4 oz. of albumen, 4 oz. of fat, and 16 oz. of starch. The following table shows the approximate amounts of these elements in various foods:

Food		Per cent of Albumen	Per cent of Fat	Per cent of Starch
I.	Bread	9	1	54
	Potatoes	2	1	20
	Rice	8	1	80
	Macaroni	13	1	74
	Tapioca	none	none	88
II.	Beef	15	15	none
	Mutton	14	37	none
	Pork	13	26	none
	Peas	25	1	62
	Beans	22	2	60
III.	Olive Oil	none	98	none
	Butter	1	83	none
	Cheese	3	71	none
	Cream	2	18	4
	Eggs	13	9	none

How many ounces each of bread, beef, and cheese are necessary to furnish the standard day's ration?

Solution. 1. Let x, y, z be the number of ounces of each required. According to the table these quantities will furnish:

2. $.09\,x + .15\,y + .03\,z$ oz. of albumen,

3. $.01\,x + .15\,y + .71\,z$ oz. of fat, and

4. $.54\,x + 0\,y + 0\,z$ oz. of starch.

Since the standard ration is to be furnished,

5. $.09\,x + .15\,y + .03\,z = 4.$

6. $.01\,x + .15\,y + .71\,z = 4.$

7. $.54\,x = 16.$

8. $\therefore x = 29.6.$

Substituting this value of x in 5 and 6, and multiplying both equations by 100, to clear the left members of fractions,

9. $$15y + 3z = 133.6.$$

10. $$15y + 71z = 370.4.$$

Solving (9) and (10), $z = 3.5$; $y = 8.2$.

Therefore 29.6 oz. of bread, 8.2 oz. of beef, and 3.5 oz. of cheese would furnish the standard daily ration.

Negative values would show that the foods selected cannot furnish the standard ration. If one food is taken from each of the three groups in the table, the combinations will be more likely to furnish the standard ration.

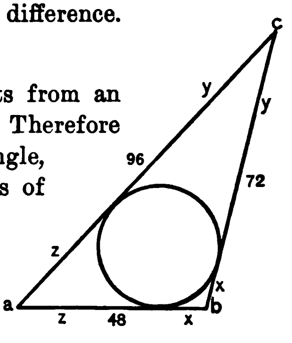
Find which of the following combinations of foods can make a standard ration and the number of ounces of each food required:

21. Potatoes, beef, and cheese.
22. Rice, pork, and cream.
23. Macaroni, peas, and cream.
24. Tapioca, beans, and eggs.
25. Bread, beans, and olive oil.
26. Bread, butter, and beef.
27. Cheese, beans, and tapioca.
28. Rice, mutton, and cream.

29. It is known that the sum of the three angles of any triangle is 180°. In a certain triangle the difference between the first angle and the second is 10°, and between the second and the third is 25°. Find the number of degrees in each angle.

30. The sum of the dimensions of a rectangular box is $13\frac{1}{2}$ feet; the height equals one half the sum of the length and breadth and also equals twice their difference. Find the dimensions.

31. It is known that the tangents from an external point to a circle are equal. Therefore when a circle is inscribed in a triangle, as in the figure, there are three sets of equal tangents, as x, x; y, y; z, z. What is the sum of x and y? Of z and y? Of x and z? Determine the length of x, of y, and of z.



277. Equations with more than Three Unknowns. The solution of a particular example will serve to indicate the method to be used when there are more than three unknowns.

EXAMPLE

Solve:
$$\begin{cases} w+x+y+z=8, & (1)\\ 2w-\frac{1}{2}x+\frac{1}{3}y-z=1, & (2)\\ 3w-2x-y+2z=17, & (3)\\ 5w-x-2y-3z=-9. & (4)\end{cases}$$

To eliminate z:

Adding (1) and (2), $\quad 3w+\frac{1}{2}x+\frac{4}{3}y=9. \quad (5)$

Subtracting twice (1) from (3), $\quad w-4x-3y=1. \quad (6)$

Adding three times (1) and (4), $\quad 8w+2x+y=15. \quad (7)$

To eliminate x:

Subtracting (7) from 4 times (5), $\quad 4w+\frac{13}{3}y=21. \quad (8)$

Adding twice (7) to (6), $\quad 17w-y=31. \quad (9)$

To solve for w:

Substituting y from (9) into (8), $\quad 4w+\frac{13}{3}(17w-31)=21. \quad (10)$

Solving (10), $\quad 233w=466$, and $w=2. \quad (11)$

To solve for the remaining unknowns:

Substituting $w=2$ in (9), $\quad y=3. \quad (12)$

Substituting $y=3$, $w=2$ in (7), $\quad 16+2x+3=15. \quad (13)$

Solving (13), $\quad x=-2. \quad (14)$

Substituting $x=-2$, $y=3$, $w=2$ in (1), $\quad 2-2+3+z=8. \quad (15)$

Solving (15), $\quad z=5. \quad (16)$

TEST. (1) Used in (15),

(2) $2-\frac{1}{2}\cdot(-2)+\frac{1}{3}\cdot 3-5=1.$

(3) $3\cdot 2-2\cdot(-2)-3+2\cdot 5=17.$

(4) $5\cdot 2-(-2)-2\cdot 3-3\cdot 5=-9.$

In elementary algebra little emphasis should be placed on problems with more than three unknowns; in later mathematics better and more general methods are given for dealing with them.

To later mathematics, likewise, belongs the discussion and proof of the assumption which we have tacitly made that the processes used in the solution of systems of simultaneous equations lead to equations equivalent to those given.

WRITTEN EXERCISES

Solve:

1. $\frac{1}{3}x + 3y = 23,$
$x + \frac{1}{4}z = 8,$
$y + 3z = 31,$
$x + w + y = 22.$

2. $w + x + y = 9,$
$x + y + z = 9,$
$y + z + w = 9,$
$z + w + x = 9.$

SUGGESTION. Add the equations, divide by 3, and subtract each from the result.

3. $3w - x + y - 2z = -y,$
$w - 3x - 2y + z = -9,$
$5w + 2x - 3y + 2z = 8,$
$2w - 2x + 4y - 3z = -2.$

4. $w - x + y = 6,$
$x - y + z = -4,$
$y - z + w = 5,$
$z - w + x = -6.$

5. $w + y + z = 2x,$
$w + x + 2 = 3y,$
$w + x + y = 42,$
$w - x - y = w.$

6. $7x - 3y = 1,$
$112 - 7w = 1,$
$4z - 7y = 1,$
$19x - 3w = 1.$

7. $v + w = 6,$
$2 - w = 8,$
$3z - 2v = 10,$
$2x - y = 12,$
$y + z = 14.$

8. $x - 2y + 32 = 10,$
$2w - 3v + 4x = 13,$
$4y - 6z + 3w = -8,$
$3z - 7w + 5v = 6,$
$5v - x + 3y = 14.$

9. $\frac{1}{x} - \frac{1}{y} = \frac{1}{6},$
$\frac{1}{y} - \frac{1}{z} = \frac{1}{12},$
$\frac{1}{x} + \frac{1}{z} = \frac{3}{4}.$

10. $\frac{a}{x} + \frac{b}{y} = m,$
$\frac{b}{y} + \frac{c}{z} = n,$
$\frac{a}{x} + \frac{c}{z} = p.$

11. $x + y + z = 1,$
$5x + y - .22 = .5,$
$2x + 3y + 3z = 1.$

SUMMARY

1. A *general form* for an equation of the first degree in two unknowns is:

$$ax + by = e.$$

A *general form* for two such equations is:

$$\begin{cases} ax + by = e, \\ cx + dy = f. \end{cases} \qquad \text{Sec. 266.}$$

2. The *general solution* of these equations is:

$$x = \frac{de - bf}{ad - bc}, \qquad y = \frac{af - ce}{ad - bc}. \qquad \text{Sec. 267.}$$

3. *Solution of fractional equations.*

(1) In case of fractional equations, when the unknowns occur only in monomial denominators, it is best not to clear of fractions. Sec. 270.

(2) If the unknowns occur in polynomial denominators, it is generally best to clear of fractions. Sec. 271.

(4) The solution of two simultaneous equations of the first degree in two unknowns is represented graphically by the point of intersection of the graphs of the equations. Sec. 274.

(5) To solve three linear equations with three unknowns, eliminate one unknown from any pair of the equations and the same unknown from any other pair; two equations are thus formed which involve only two unknowns and which may be solved by the methods for solving such equations.

Four or more equations with four or more unknowns may be solved similarly.

REVIEW

WRITTEN EXERCISES

Solve and test:

1. $x + y = 27,$
   $x - y = 17.$

2. $3x + 5y = 19,$
   $7x - 4y = 13.$

3. $\frac{1}{3}(7 + x) = \frac{1}{5}(9 + y),$
   $\frac{1}{7}(11 + x + y) = \frac{1}{3}(9 + y).$

4. $bx + ay = b,$
   $ax - by = a.$

5. $\frac{x}{a} + \frac{y}{b} = 1,$
   $\frac{x}{b} - \frac{y}{a} = 1.$

6. $\frac{x}{8}+\frac{y}{4}=11,$
$\frac{5x}{6}-\frac{3y}{8}=12.$

7. $\frac{x}{a}+\frac{y}{b}=7,$
$\frac{x}{2}+\frac{y}{3}=2a+b.$

8. $3x+5y=1,$
$4x+6y=3.$

9. $2x-9y=-1,$
$5x-24y=2.$

10. $\frac{1}{x}+\frac{1}{y}=\frac{7}{12},$
$\frac{5}{x}-\frac{3}{y}=\frac{1}{4}.$

11. $2(3x+1)-3(4y-26)=2,$
$3(x-5)+2(y-14)=6.$

12. $2p+3w=1,$
$5p+7w=6.$

13. $4h+7v=9,$
$2h+5v=6.$

14. $2x+6y+7z=4,$
$3x+8y+9z=-2,$
$4x+9y+10z=1.$

15. $4x+11y+2z=-41,$
$15x+39y+7z=\frac{291}{2},$
$23x+56y+10z=-209.$

16. $x+y+z=4,$
$2x+3y-z=1,$
$3x-y+2z=1.$

17. $6x+2y+8z=1,$
$9x+8y+11z=-1,$
$15x+12y-3z=-1.$

18. $x-2y+3z-u=5,$
$y-2z+3u-x=0,$
$z-2u+3x-y=0,$
$u-2x+3y-z=0.$

19. Solve for a and b:

$$ac+bq=d,$$
$$ad+5b=q.$$

Solve the equations of 19 for:

20. a and q,
21. c and d,
22. c and q,
23. d and q.

24. A man had two sums invested, one at 4%, the other at 5%, simple interest, and thus received \$500 annually. If the rates of interest had been 5% and 6% respectively, he would have received \$110 more per annum. Find the sums invested.

25. A man's investments in bonds brought him 4%, in mortgages 5%, and real estate 6% income per annum, making a total of \$1360. At a later period the rates of income on the same amounts were 5% for bonds, 6% for mortgages, and 4% for real estate, producing a total annual income of \$1260. Still later the rates on the same amounts were 6% for bonds, 4% for mortgages, and 5% for real estate, producing a total annual income of \$1280. Find the amounts of his different investments.

26. A man can walk $2\frac{1}{2}$ miles an hour up hill and $3\frac{1}{2}$ miles an hour down hill. He walks 56 miles in 20 hours on a road no part of which is level. How much of it is up hill?

27. A tourist spent \$ 520 on a trip. If he had cut down his transportation expenses $\frac{1}{3}$, his hotel bill $\frac{1}{4}$, and his miscellaneous expenses $\frac{1}{2}$, his trip would have cost him \$ 350. If he had cut down his transportation expenses $\frac{1}{2}$, increased his hotel bills by $\frac{1}{3}$, and his miscellaneous expenses by $\frac{1}{4}$, the trip would have cost him \$ 535. Find the amount he actually spent for each of the three items.

28. A man paid \$ 225 premium for the insurance of 3 houses at 75¢ per \$ 100 of valuation. The next year he diminished the valuation of the first house by $\frac{1}{3}$, and the second by $\frac{1}{4}$, and his premium was \$ 180. The third year, using the valuations of the second year as basis, he increased the valuation of the first house by $\frac{1}{4}$, of the second by $\frac{1}{3}$, and of the third by $\frac{1}{3}$, and his premium amounted to \$ 210. Find the valuation of each house the first year.

29. What values have a mark and a ruble in our money if 38 rubles are worth 14 cents less than 75 marks, and if a dollar and a ruble are together worth $6\frac{1}{3}$ marks?

30. Three thousand dollars was given annually to a college to provide annual scholarships of grades a, b, c. When two of grade a, six of grade b, and one of grade c were granted, the gift was just sufficient; similarly, when four of grade a, two of grade b, and two of grade c were granted, and also when one of grade a, five of grade b, and five of grade c were granted, the gift was sufficient. What was the value of each scholarship?

31. The number of adults and the number of children likely to attend a certain entertainment were estimated in advance; the sum to be raised by the entertainment was \$ 550; if the admission for adults was fixed at 40 cents and that for children at 30 cents, the estimated receipts would lack \$ 90 of the required amount; but if the admission was fixed at 50 cents for adults and 25 cents for children, the exact sum would be raised. How many of each were expected to attend?

32. A tank contains 20 gal. of water, and water flows in at the rate of 5 gal. per minute. At the same time a second tank contains 50 gal. of water, and water flows in at the rate of 2 gal. per minute. Construct a graph to represent the amount of water in the first tank for each minute from 0 to 15. In the same figure draw a graph to represent the amount of water in the second tank for each minute. From the graph read at what time the two tanks will contain equal amounts of water and what the amount is. Verify by solving algebraically.

33. A merchant pays $10 rent weekly. His profits on his sales average 20%. Represent graphically his net profits corresponding to weekly sales ranging from 0 to $200. A second merchant sublets part of his store for $5 per week more than his own rental, but he makes only 10% average profits on his sales. In the same figure represent his total profits for sales ranging from 0 to $200. From the graph read the amount of sales for which both merchants make the same net profit. Verify by solving algebraically.

34. Three of the longest tunnels in the world, the St. Gotthardt, Mont Cenis, and Arlberg, are together 37,672 meters in length. If each were 1617.45 meters longer, the 1st and 2nd would be in the ratio of 10 to 7, and the 2nd and 3rd in the ratio of 7 to 6. Find their lengths.

35. If tin and lead lose, respectively, $\frac{5}{37}$ and $\frac{2}{23}$ of their weights when weighed in water, and a 60-pound mass of lead and tin loses 7 lb., find the weight of the tin in this mass.

36. What are the sides of a rectangle such that: (*a*) the area is not changed if the base is diminished by 2 and the altitude increased by 2; (*b*) the area is increased by 10, if both base and altitude are increased by 1?

37. In a certain mill there are two rates of pay, one $1.50 a day, the other higher. The total paid in wages each day is $350. An equal assessment made by a labor union to raise $200 requires $1.00 from each man receiving $1.50 a day, and half of one day's pay from every man receiving more. How many men receive $1.50 a day?

SUPPLEMENTARY WORK

Equations with Two Unknowns

ADDITIONAL EXERCISES

1. A certain number is written with two digits; twice the tens' digit plus the units' digit makes 9; when the digits are interchanged, the number formed is 27 greater than the given number. Find the original number.

SOLUTION. 1. Let $x =$ the tens' digit, and $y =$ the units' digit.

2. Then the number is $10x + y$, and $10y + x$ is the number with digits interchanged.

3. Then, $2x + y = 9$; and

4. By the conditions of the problem, $10y + x - 27 = 10x + y$.

5. Simplifying (4), $y - x = 3$.

6. Subtracting (5) from (3), $3x = 6$.

7. Solving (6), $x = 2$.

8. Substituting $x = 2$ in equation (5), $y = 5$.

$\therefore$ the number is $10 \cdot 2 + 5$, or 25.

TEST. $2 \cdot 2 + 5 = 9$. $52 = 25 + 27$.

2. In a certain number of two digits the sum of three times the tens' digit and twice the units' digit is 36; if the digits are interchanged, the number formed is 27 greater than the original number. What is the original number?

3. The sum of two numbers is 1.6 and their difference is .2. What are the numbers?

4. The difference between two numbers is 30, and the less is $\frac{2}{5}$ of the greater. What are the numbers?

5. Two numbers are such that if the first is increased by 14, the result is twice the second; if the second is diminished by 12, the result is $\frac{1}{3}$ the first. What are the numbers?

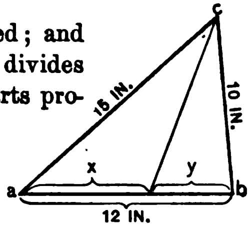
6. A farmer has feed enough to last his oxen a certain number of days. If he were to sell 75 oxen, his feed would last 20 days longer. If, however, he were to buy 100 oxen, his feed would last 15 days less. How many oxen has he and for how many days has he feed enough?

SUGGESTION. Let x = number of oxen,
y = number of days.

Then $(x-75)(y+20)=xy.$
$(x+100)(y-15)=xy.$

7. The dimensions of a rectangular floor are such that, if it were 3 ft. longer and 2 ft. wider, its area would be 64 sq. ft. greater; and if it were 2 ft. longer and 3 ft. wider, its area would be 68 sq. ft. greater. Find its dimensions.

8. The angle c in the figure is bisected; and it is known that the bisector of an angle divides the opposite side of the triangle into parts proportional to the adjacent sides; what is the ratio of x to y in the figure? What is the sum of x and y? Find the length of each segment.



9. A person leaves an estate worth \$13,000; some of it is willed to a college, and 12 times as much to an eldest son, whose share is $1\frac{1}{2}$ times as much as that of each of his 2 brothers and $1\frac{2}{3}$ times that of each of 5 sisters. Find the amount left the college.

10. Divide $x^4+x^3+ax^2+bx-3$ by x^2+2x-3, and find what values a and b must have in order that there be no remainder.

SUGGESTION. The coefficient of x and the absolute term in the remainder must both be zero. Solve the resultant equations for a and b.

11. The sum of two numbers is a^2 and their difference is b^2; what are the numbers? What are the numbers if $a=b$?

12. A band of smugglers found a cave, which would exactly hold the cargo of the boat, namely, 13 bales of silk and 33 casks of rum. While they were unloading a revenue cutter was sighted, and they sailed away, leaving 9 casks and 5 bales; these filled only one third of the cave. How many bales would the cave hold? How many casks?

13. A and B play at a game with counters. In the first game A loses as many counters as B has; in the second game B loses as many counters as A then has; at the end of the

second game A has 14 counters and B has 4. How many had each at first?

14. A company at a tavern, when they came to pay, found that if the same bill were divided among three persons more, the amount would be one shilling less per person; and, if it were divided among two persons fewer, it would be one shilling more per person. Find the number of persons in the original company, and the amount of the bill. (Saunderson's *Algebra*, 1740.)

15. One person says to another, "If you give me three of your coins, I shall have as many as you." The second person replies, "If you give me three of yours, I shall have twice as many as you have." Find the numbers that each has. (Ozanam's *Algebra*, 1702.)

16. A coach set out from Cambridge to London with four more passengers outside than within. Seven outside passengers could travel at 2 shillings less expense than 4 inside passengers. The fares of all the passengers amounted to 180 shillings. At the end of half the journey the coach took up 1 more inside and 3 more outside passengers; these paid $\frac{2}{15}$ as much as the others. Required the number of passengers and the fare of each. (Bland's *Algebraical Problems*, 1816.)

17. Three casks together contain 79 gallons; the second contains 3 gallons more than $\frac{1}{2}$ as much as the first, and the third contains 7 gallons less than the second. How many gallons are there in each? (From a 14th century manuscript.)

18. Seven years ago a man was 4 times as old as his son; 7 years hence he will be only double his age. Find the age of each. (Simpson's *Algebra*, 1767.)

Equations with Three Unknowns

ADDITIONAL EXERCISES

1. Find an expression of the form $ax^2 + bx + c$ whose value is 6 when $x = 2$. Whose value is 3 when $x = -1$, and 10 when $x = 4$.

2. The sum of the reciprocals of three numbers is $\frac{13}{6}$; the difference between the reciprocals of the first and second equals that between the reciprocals of the second and third. The third number is twice the first. Find the numbers.

3. Find three numbers such that the difference between the reciprocals of the first and second is $\frac{1}{6}$, between the reciprocals of the first and third is $\frac{1}{4}$, and the sum of the reciprocals of the second and third is $\frac{7}{12}$.

4. The sum of the reciprocals of the first and third of three numbers is twice the reciprocal of the second; the reciprocal of the third is four times that of the first; the sum of the reciprocals of the first and second is 7. Find the numbers.

5. A man has three debtors, of whom A and B together owe him 60 pounds, A and C 80 pounds, and B and C 92 pounds. How much did each one owe? (Saunderson's *Algebra*, 1740.)

6. A vessel filled with water has three orifices, A, B, C. If all three are opened, it is emptied in 6 hr.; through B alone it is emptied in $\frac{2}{3}$ of the time that it would take A alone; and the time through C is 5 hours greater than through B. In what time is the vessel emptied through each orifice alone? (Bossut's *Algebra*, 1773.)

7. The price of a house is 100 dollars. A could pay for it if he had half of B's money in addition to his own; B could pay for it if he had one third of C's; and C could pay for it if he had one fourth of A's money. How much had each? (Euler's *Algebra*, 1770.)

8. Three brothers, A, B, C, at a family reunion were discussing their ages. C said to A, "Thirty years ago my age was double yours." Then B said to A, "Twenty-three years ago *my* age was double yours." If C's present age exceeds B's by three years and B's exceeds A's by seven years, find the age of each.

CHAPTER XVI

QUADRATIC EQUATIONS

278. Quadratic Equations. Equations of the second degree are called **quadratic equations.** Sec. 239, p. 198.

For example, $x^2 = 16$, $x^2 - 3x = 0$, and $x^2 - 5x + 6 = 0$ are quadratic equations with one unknown, x.

The present chapter treats quadratic equations with one unknown.

279. The way in which quadratic equations arise in practice is illustrated in the following exercises.

WRITTEN EXERCISES

Translate each statement into an equation:

1. The product of a certain number and the number increased by 3 is 70.

2. Three less than one half of a certain number is 47 less than the square of the number.

3. Twice a certain number multiplied by one more than 4 times the number is 3.

4. 3 more than a certain number multiplied by 5 more than the number is 8.

5. Let I be the estimated annual income per inhabitant in Great Britain. In the United States this income is $5 greater and the product of the two numbers is 1326.

6. In a rectangular orchard the number of trees in a row is 5 more than the number of rows, and there are in all 150 trees.

7. The product of two consecutive integers is 132.

8. A book is estimated to contain 10,500 words. The average number of words per page is 5 greater than the number of pages.

9. The area of a rectangle whose length is three times its height is 75 sq. in.

10. In the United States recently the number of newspapers sent by mail per inhabitant was 180 more than the square of $\frac{1}{40}$ of the number.

11. If the number of hundred thousand inhabitants of Massachusetts recently be multiplied by the number increased by 3, the result is 868.

12. In the case of a body falling from rest, the distance d fallen in the time t is one half of a fixed number g (the constant of gravity) times the square of the time.

280. General Form. The *general form* of the quadratic equation is

$$ax^2 + bx + c = 0,$$

in which a, b, c, are any known numbers, except that a may not be zero.

For example:

	a	b	c
$x^2 + 5x - 3 = 0,$	1	5	-3
$3x^2 - x + 5 = 0,$	3	-1	5
$x^2 + 7x = 0,$	1	7	0
$4x^2 - 12 = 0,$	4	0	-12
$x^2 = 0,$	1	0	0

281. It is often necessary to simplify equations apparently involving x^2 to see whether or not a is zero; that is, whether or not the equations are really quadratics.

For example:

$\frac{x^2 - 3}{5x} = \frac{x + 2}{7}$ can more readily be seen to be a quadratic equation when reduced to $2x^2 - 10x - 21 = 0$.

In this form it is apparent that $a = 2$, $b = -10$, $c = -21$.

WRITTEN EXERCISES

Reduce each equation to the type form and write the value of a; of b; of c:

1. $\frac{x^2-1}{2}=1.$
2. $3x^2-5=\frac{x-1}{2}.$
3. $\frac{x^2-1}{2}=\frac{x^2-4}{5}.$
4. $\frac{2x^2+3}{5}=\frac{x^2}{7}.$
5. $x+\frac{1}{x}=3.$
6. $12-x=\frac{x^2}{8}.$
7. $\frac{x+6}{7}=\frac{x^2}{2}.$
8. $\frac{2x-9}{3}=\frac{2x^2}{5}.$

282. Kinds of Quadratic Equations. The equation $ax^2+bx+c=0$ is said to be a **complete quadratic equation** when neither b nor c is zero; that is, when there are three terms, one containing x^2, another x, and a term without x (absolute term). In any other case it is called an **incomplete quadratic equation.**

283. A quadratic in which the first power of the unknown does not occur is called a **pure** quadratic, and one in which it occurs is called an **affected** quadratic.

Thus, $x^2+5x-2=0$ is a complete quadratic equation, while $2x^2-5=0$, and $x^2+7x=0$ are incomplete quadratics. The second equation is a pure quadratic and the others are affected quadratics.

284. Solution of Incomplete Quadratic Equations.

I. *The incomplete quadratic equation $x^2=a$ is solved by extracting the square root of both members.*

EXAMPLES

1. $x^2=4.\quad \therefore x=\sqrt{4}$ or $x=\pm 2.$

2. $3x^2=78.\quad \therefore x^2=26$ and $x=\pm\sqrt{26}.$

3. $ax^2=b.\quad \therefore x^2=\frac{b}{a}$ and $x=\pm\sqrt{\frac{b}{a}}.$

II. *The incomplete quadratic equation* $ax^2 + bx = 0$ *is solved by factoring* (Sec. 233, p. 186).

Thus, $x(ax + b) = 0$, in which $x = 0$ and $x = -\frac{b}{a}$.

One value of x in this case is always zero (Sec. 105, p. 67), and the other is the root of the linear equation $ax + b = 0$.

EXAMPLES

1. $x^2 - x = 0$. $\therefore x(x - 1) = 0$ and $x = 0$, $x = 1$.
2. $3x^2 - 10x = 0$. $\therefore x(3x - 10) = 0$ and $x = 0$, $x = \frac{10}{3}$.

WRITTEN EXERCISES

Solve the following equations:

1. $x^2 = 169$.
2. $x^2 - 121 = 0$.
3. $x^2 - 144 = 0$.
4. $x^2 - 81 = 0$.
5. $x^2 - 49 = 0$.
6. $x^2 - 625 = 0$.
7. $3x^2 - 75 = 0$.
8. $4x^2 - 100 = 0$.
9. $5x^2 - 500 = 0$.
10. $12x^2 - 1728 = 0$.
11. $121x^2 = 1089$.
12. $7x^2 - 448 = 0$.
13. $3x^2 + 6 = 5x^2 - 8$.
14. $17x^2 + x = 0$.
15. $40x^2 - 25x = 0$.
16. $9x^2 + 17 = 4x^2 - 13$.
17. $29x^2 - 30 = 10x^2 + 8$.
18. $40x^2 - 43 = 7 - 10x^2$.
19. $7x^2 - 5 = 4x^2 - 7$.
20. $3(x^2 - x) = 2x^2 - 3x + 1$.

21. The equation $d = \frac{1}{2}gt^2$ expresses the distance through which a body falls from rest in time t. Solve this equation for t; that is, express t in terms of d and g.

22. When d is expressed in feet and t in seconds, g is approximately 32. Find the value of t when $d = 16$.

23. Find the time in seconds taken by a body in falling from rest 64 ft. Also 144 ft. Also 256 ft.

24. A stone was dropped into a mine 400 ft. deep. How many seconds was it in reaching the bottom?

25. Solve $d = \frac{1}{2}gt^2$ for g; that is, express g in terms of t and d. A body falls from rest 256 ft. in 4 sec. From this find the value of g.

26. The area (a) inclosed by a circle is approximately $\frac{22}{7}$ of the area of the square whose side is the radius (r) of the circle. Assuming these areas to be equal, $a = \frac{22}{7}r^2$. Express r in terms of a. Find r when $a = \frac{242}{49}$.

27. The area of a certain circle is $\frac{22}{7}$ sq. ft. Find the length of the radius in feet.

28. Find the radius of a circle whose area is $\frac{22}{63}$ sq. in.; $\frac{22}{175}$ sq. in.; $\frac{22}{112}$ sq. yd.

29. The length of a pendulum in meters is approximately the square of the number of seconds taken for one vibration. That is, $l = t^2$. What is the time of vibration of a pendulum 4 meters long? 1 m. long? 16 m. long?

30. The kinetic energy (k) of a moving body equals one half of the product of its mass (m) and the square of its velocity (v). That is, $k = \frac{1}{2}mv^2$. Solve this equation for v. For m.

31. Find the value of v in the equation $k = \frac{1}{2}mv^2$, when $k = 200$ and $m = 4$.

32. As in the last exercise, find m when $k = 100$ and $v = 5$.

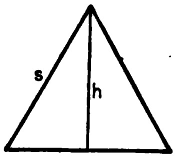
33. The velocity (v) of a body equals the square root of the quotient of twice its kinetic energy (k), divided by its mass (m). Express this relation by an equation.

34. According to Exercise 33, find the kinetic energy in pounds of a coal car whose mass is 50,000 lb., moving at a velocity of 20 ft. per sec.

35. Also, find the velocity in feet per second of a bullet of mass $\frac{1}{8}$ lb., having a kinetic energy of 900 lb.

36. Find the velocity in feet per second of an engine weighing 75 tons, and having a kinetic energy of 135,000 tons. Express this result in miles per hour.

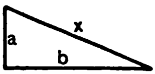
37. 4 times the square of the altitude (h) of an equilateral triangle equals 3 times the square of a side s. Express this by an equation. Solve the equation for s. Also for h in terms of s.



38. Find the altitude of an equilateral triangle whose side is 20 in., using 1.7321 as $\sqrt{3}$.

39. Find the side of an equilateral triangle whose altitude is $9\sqrt{3}$ in.

40. The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides. Express this relation in the form of an equation, using the letters in the figure.



41. Find the length of the hypotenuse of a right triangle, the other two sides of which are 3 ft. and 4 ft. Also of one whose other two sides are 15 ft. and 20 ft.

42. Solve the equation $c^2 = a^2 + b^2$ for a; for b.

43. Find b in Exercise 42, if $c = 25$ and $a = 15$.

Similarly, determine the numbers to fill the blanks. To simplify calculation, see Sec. 133, p. 85.

	44.	**45.**	**46.**	**47.**	**48.**	**49.**	**50.**
a.	15	7	40	45	—	208	44
b.	—	24	—	—	171	—	117
c.	17	—	41	53	221	233	—

285. Solution of Complete Quadratic Equations. The following example shows a general method for solving quadratic equations with one unknown:

EXAMPLE

Solve: $$x^2 + 8x + 7 = 0. \qquad (1)$$

Transposing the absolute term, $$x^2 + 8x = -7. \qquad (2)$$

Making the left member a square by adding 16 to *both* members, $$x^2 + 8x + 16 = 9. \qquad (3)$$

$$\therefore (x + 4)^2 = 9. \qquad (4)$$

Extracting the square root of both members, $$x + 4 = \pm 3. \qquad (5)$$

$$\therefore x = -1 \text{ or } -7. \qquad (6)$$

The process consists of two main parts:

(1) *Making the left member a square while the right member does not contain the unknown quantity.*

This is called completing the square.

It is based upon the relation $(x + a)^2 = x^2 + 2ax + a^2$ (Sec. 119, p. 79) in which it appears that the last term, a^2, is the square of one-half of the coefficient of x.

(2) *Extracting the square roots of both members and solving the resulting linear equations.*

WRITTEN EXERCISES

Solve completely:

1. $x^2 + 8x = 9$.
2. $x^2 + 4x = 12$.
3. $x^2 + 12x = -11$.
4. $x^2 - 8x = 9$.
5. $x^2 + 10x = 11$.
6. $x^2 + 5x = \frac{11}{4}$.
7. $x^2 - 20x = -75$.
8. $x^2 - 6x = 7$.
9. $z^2 - 40z = 41$.
10. $w^2 - w = \frac{3}{4}$.
11. $t^2 - 2t = 8$.
12. $u^2 - u = -\frac{1}{4}$.
13. $v^2 + 3v = \frac{-5}{4}$.
14. $s^2 - 18s = 19$.

286. Square roots which cannot be found exactly should be indicated.

EXAMPLE

Solve: $x^2 - 6x + 3 = 0$. *(1)*

Rearranging and completing the square: $x^2 - 6x + 9 = 6$. *(2)*

$\therefore (x - 3)^2 = 6$. *(3)*

$\therefore x - 3 = \pm\sqrt{6}$. *(4)*

$\therefore x = 3 + \sqrt{6}$ and $3 - \sqrt{6}$. *(5)*

WRITTEN EXERCISES

Solve:

1. $y^2 - y = 1$.
2. $x^2 + x = 5$.
3. $x^2 - 6x = -1$.
4. $x^2 - 36x = -3$.
5. $x^2 + 15x = 7$.
6. $x^2 - 40x = 20$.

7. $x^2 - 7x = \frac{3}{2}$.
8. $x^2 - \frac{1}{4}x = \frac{1}{2}$.
9. $z^2 - 16z = 5$.
10. $t^2 - 2t = 6$.
11. $u^2 - u = 1$.
12. $x^2 + 13x = 9$.
13. $x^2 + 6x = -4$.
14. $x^2 - 8x = -8$.
15. $x^2 + 13x = \frac{3}{4}$.
16. $x^2 + 15x = \frac{7}{2}$.
17. $x^2 - 8x = 3$.
18. $m^2 + 8m = 4$.
19. $x^2 + 10x = 1$.
20. $x^2 - 5x + 6 = 0$.
21. $x^2 - 8x = 3$.
22. $x^2 + 10x = 1$.
23. $p^2 - 6p = -1$.
24. $x^2 - 36x = -13$.
25. $x^2 + 25x = 7$.
26. $x^2 - 40x = 30$.
27. $x^2 - 9x = \frac{3}{2}$.
28. $x^2 - \frac{3}{4}x = \frac{1}{2}$.
29. $z^2 - 16z = 9$
30. $t^2 - 8t = 6$.
31. $u^2 - u = 5$.
32. $x^2 + 33x = 9$.
33. $m^2 - 4m = 1$.
34. $w^2 + 5w + 6 = 0$.
35. $t^2 + 9t + 20 = 0$.
36. $v^2 - v - 20 = 0$.
37. $x^2 - x - 42 = 0$.
38. $x^2 - 5x - 84 = 0$.
39. $u^2 + 19u + 84 = 0$.
40. $z^2 - 9z + 14 = 0$.

287. When the coefficient of x^2 is not unity, the equation must be divided by that coefficient before using the method above to complete the square.

EXAMPLE

Solve: $$5x^2 + 7x - 2 = 0. \qquad (1)$$

Dividing by 5, $$x^2 + \frac{7}{5}x - \frac{2}{5} = 0. \qquad (2)$$

Completing the square, $$x^2 + \frac{7}{5}x + \frac{49}{100} = \frac{2}{5} + \frac{49}{100}. \qquad (3)$$

Or, $$\left(x + \frac{7}{10}\right)^2 = \frac{89}{100}. \qquad (4)$$

Extracting the square root, $$x + \frac{7}{10} = \pm\frac{1}{10}\sqrt{89}, \qquad (5)$$

and $$x = -\frac{7}{10} \pm \frac{1}{10}\sqrt{89}. \qquad (6)$$

Throughout this chapter, the square roots are to be indicated only, unless the expression under the radical sign is a perfect square.

All results should be tested by substitution.

WRITTEN EXERCISES

Solve and test:

1. $4x^2+6x+3=0.$
2. $9x^2+15x+6=0.$
3. $4x^2-2x-2=0.$
4. $9x^2+3x-6=0.$
5. $25z^2-15z+4=0.$
6. $3x^2-7x-20=0.$
7. $4x^2+12x-55=0.$
8. $9w^2+6w-35=0.$
9. $9v^2-39v+22=0.$
10. $4y^2-12y=91.$
11. $16t^2-8t=15.$
12. $4z^2+20z=-21.$
13. $6x^2+x=12.$
14. $6x^2=-5x+4.$
15. $12x^2=5x+2.$
16. $9x^2=3x-5.$

Clear each of the following of fractions and solve the resulting equation:

17. $\dfrac{120}{x+3}=\dfrac{120}{x}-2.$
18. $\dfrac{5x}{x+4}-\dfrac{3x-2}{2x-3}=2.$
19. $x-\dfrac{x^3-8}{x^2+5}=2.$
20. $\dfrac{1}{3}+\dfrac{1}{3+x}+\dfrac{1}{3+2x}=0.$
21. $\dfrac{x+22}{3}-\dfrac{4}{x}=\dfrac{9x-6}{2}.$
22. $\dfrac{x+2}{x-1}-\dfrac{4-x}{2x}=2\frac{1}{3}.$
23. $\dfrac{48}{x+3}=\dfrac{165}{x+10}-5.$
24. $\dfrac{x}{x+1}+\dfrac{x+1}{x}=\dfrac{13}{6}.$
25. $\dfrac{2x}{x-4}+\dfrac{2x-5}{x-3}=\dfrac{25}{3}.$
26. $\dfrac{3x-7}{x}+\dfrac{4x-10}{x+5}=\dfrac{7}{2}.$
27. $\dfrac{4x+7}{19}+\dfrac{5-x}{3+x}=\dfrac{4x}{9}.$
28. $\dfrac{12}{5-x}+\dfrac{4}{4-x}=\dfrac{32}{x+2}.$

29. The perimeter of a rectangular field is 100 yd. and its area is 600 sq. yd. Find its length and breadth.

SOLUTION. 1. Let x be the length of the field.

2. Then $\frac{600}{x}$ is its width, and

3. $2x + \frac{2 \cdot 600}{x}$ is its perimeter, being twice the sum of its sides.

4. $\therefore 2x + \frac{2 \cdot 600}{x} = 100$, by the conditions of the problem.

5. $\therefore x^2 - 50x + 600 = 0$, simplifying (4).

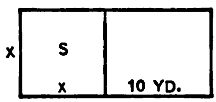
6. $\therefore (x - 20)(x - 30) = 0$, factoring (5).

7. $\therefore x = 20$, $x = 30$, solving (6).

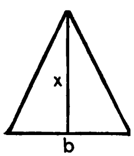
8. If 20 yd. be taken as the length, the width is 30 yd., by step (2).

9. If 30 yd. be taken as the length, the width is 20 yd., by step (2).

30. The area of the whole plot shown in the diagram is 96 sq. yd. What is the length of a side of the square (s)?



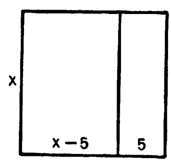
31. The sum of two unequal sides of a rectangular court is 19 yd.; the sum of the areas of the squares on these two sides is 181 sq. yd. What are the dimensions of the court?



32. A triangle whose area is 200 sq. yd. has its altitude equal to its base. Find the base of the triangle, using the formula, area = $\frac{1}{2}$ base $\times$ altitude.

33. The breadth of a room is 4 ft. more than its height and 20 ft. less than its length. The cost of cleaning and painting the side walls was $ 20.70, at $2\frac{1}{2}$¢ per square foot. Find the dimensions of the room.

34. A partition is built 5 ft. from one side of a square room as shown in the diagram; the area of the floor remaining is 24 sq. yd. What are the dimensions of the floor?



35. One side of a rectangle is $\frac{3}{4}$ as long as the other; if 10 ft. be added to the shorter side and 10 ft. be subtracted from the longer side, the area will not be changed. Find the dimensions of the original rectangle.

36. If there are s subscribers in a telephone exchange, the total number of different connections of any subscriber with any other is $\frac{s(s-1)}{2}$. If in an exchange the total number of different connections is 3240, find the number of subscribers.

37. A builder used steel bars weighing 120 lb. each for a certain purpose. By changing the mode of support, he found that he could get the same service from bars weighing 2 lb. less per running foot, but 2 ft. longer than the original bars. The new bars also weighed 120 lb. Find the length of the original bars.

38. A hall is lighted by a certain number of incandescent electric lights and 5 fewer of gas mantles. The candle power of each of the former is 70 greater than that of each mantle. The total candle power of the gas mantles is 500, and that of the electric lights 1800. How many lamps of each sort?

SUMMARY

I. **Definitions.**

1. Equations of the second degree are called *quadratic equations.* Sec. 278.

2. The *general form* of the quadratic equation in one unknown is $ax^2 + bx + c = 0$, in which a, b, c, are known numbers. Sec. 280.

3. The equation $ax^2 + bx + c$ is said to be a *complete* quadratic equation, if neither a, b, nor c is zero; otherwise, it is *incomplete;* if b is zero it is a *pure quadratic.* Quadratic equations that are not pure are sometimes called *affected quadratics.* Sec. 282, 283.

II. **Solution.**

1. The incomplete quadratic equation $x^2 = a$ is solved by extracting the square roots of both members. Sec. 284.

2. The incomplete quadratic equation $ax^2 + bx = 0$ is solved by factoring. Sec. 284.

3. Any quadratic equation can be solved by completing the square, extracting the square roots of both members, and solving the resulting linear equations. Sec. 285.

REVIEW

WRITTEN EXERCISES

Solve the following equations:

1. $x^2 + 7x = 8$.
2. $3x^2 = 48$.
3. $x^2 + 25x = -100$.
4. $(3x + 4)^2 = 96$.
5. $x^2 - 25x + 144 = 0$.
6. $x^2 + 3x - 28 = 0$.
7. $x^2 - 13x = 68$.
8. $x^2 - 12x + 27 = 0$.
9. $x^2 + 111x = 3400$.
10. $\dfrac{2x^2 + 4}{9} = x + 1$.
11. $5x^2 + 13x = 370$.
12. $\dfrac{6}{x} + 4 = 2x$.
13. $x^2 + \frac{10}{3}x = 19$.
14. $5 - 3x + \frac{1}{4}x^2 = 0$.
15. $x^2 - 7x = 0$.
16. $y^2 - ay = c$.
17. $x^2 - 12 = 30 + x$.
18. $3z^2 - 2z = 1$.
19. $y^2 + 4ay - 2 = 0$.
20. $t^2 + 3t - 6 = 0$.
21. $13x^2 - 39x = 0$.
22. $a^2x^2 - abx = 0$.
23. $(a + b)x^2 + cx = 0$.
24. $m^2x^2 - (m + r)x = 0$.
25. $ax^2 - bx = cx^2 + dx$.
26. $\frac{1}{2}x^2 = 14 - 3x^2$.
27. $x^2 + 5 = \frac{10}{3}x^2 - 16$.
28. $12x^2 - 3x = 7x^2 + 2x$.
29. $10x^2 + 5x = 15x^2 + 9x$.
30. $(a + b)^2x^2 + (a + b)x = 0$.

31. The perimeter of a rectangular field is 200 ft.; and its area is 2400 sq. ft. Find its length and breadth.

32. The area of a rectangular field is 2000 sq. ft. and its length is 10 ft. more than its breadth. Find its dimensions.

33. A miller bought \$450 worth of wheat. A month later, the price of wheat was 10¢ per bu. lower, and he could have bought 50 bu. more for \$450. At what price did he buy it?

34. After Mr. A had lived in his house 4 months longer than the number of dollars monthly rental that he paid for his house, he had paid altogether \$320 rent. How much was the monthly rental?

35. If d is the diagonal of a square of side s, then $d^2 = 2s^2$. Solve this equation for d. For s.

36. In Exercise 35, when $s = 4$, find d, taking 1.414 as the square root of 2.

37. The volume (v) of a cylinder is the product of the area of the base (πr^2) and the height (h). That is, $v = \pi r^2 h$. Solve this equation for r. For h.

38. Using $\frac{22}{7}$ for π in $v = \pi r^2 h$, find the radius of the base of a cylinder, if its volume is $\frac{22}{112}$ sq. ft. and its altitude 4 ft.

39. If water is confined by a vertical wall, its pressure on the wall is $\frac{1}{2}$ the number of square feet in the area of the part of the wall under water times the weight of 1 cu. ft. of water. Express this law of pressure by an equation, using p for pressure, l for length, and h for height, and 62.5 lb. as the weight of 1 cu. ft. of water.

40. What is the pressure on a square water gate entirely under water if each side is 10 ft.?

41. What is the height of a square water gate entirely under water against which the pressure of water is 500 lb.?

42. The length of a vertical dam is 200 ft., and the pressure of water against it is 20,000 lb. What is the height of the part under water?

43. Given $m = \frac{gv^2}{c^2}$. Express v in terms of the other letters.

44. In the preceding exercise, express c in terms of the other letters.

45. A room is 1 yd. longer than it is wide; at 75¢ per square yard, a covering for the floor of the room costs \$31.50. Find the dimensions of the floor.

46. The length of a room is twice its height, and the breadth is 6 ft. more than the height. At 10¢ per square foot it costs $72 to decorate the side walls of the room (no allowance for openings). Find the dimensions of the room.

47. The cost of decorating a certain square ceiling is $45. If a second square ceiling of side 5 yd. longer were decorated at the same rate, the cost would be $80. Find the dimensions of the first ceiling.

48. Forty-two dollars are to be divided equally each year among all the pupils of a certain room who have not been absent during that year. The second year there were two winners fewer than there were the first year, and each received 50¢ more than the prize winners of the previous year. How many of the latter were there?

49. A restaurant keeper paid out $10 for the raw materials cooked and served on Monday. Tuesday he paid out $12.60 and served 10 persons more than on Monday, at a cost of 1¢ more per person. How many were served on Monday?

50. A public library spends $75 monthly for new books. In April it bought 25 books more than in March, and the average cost per book was 25¢ less. How many books were bought in March?

51. Solve the equations obtained in Exercises 1–12, pp. 250 and 251.

CHAPTER XVII

RADICALS

DEFINITIONS AND PROPERTIES

288. Rational Numbers. Integers and other numbers expressible as the quotient of two integers are called **rational numbers.**

Thus, 5 and .2, which is expressible as $\frac{2}{10}$, are rational numbers.

289. Irrational Numbers. Any number not rational is called an **irrational number.**

Thus, $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{10}$, $\frac{1}{\sqrt{5}}$, $1 + \sqrt{3}$, $\sqrt{2} - \sqrt{3}$, π, are irrational numbers.

290. An indicated root of any number is called a **radical.**

Thus, $\sqrt{5}$, $\sqrt[3]{8}$, $\sqrt{\frac{a}{3b}}$, $\sqrt{a + x^2}$, are radicals.

In the present chapter all roots that cannot be exactly extracted by inspection are indicated. Methods for finding approximate numerical values of certain roots are given later.

291. Surd. An irrational number that is an indicated root of a rational number is sometimes called a **surd.**

Thus, $\sqrt{2}$, $\sqrt{5}$, $\sqrt[3]{7}$ are surds.

292. An expression involving one or more radicals is called a **radical expression.**

Thus, $5 + 2\sqrt{3}$, $\frac{4}{\sqrt{x}} - 1$, $\frac{8 + \sqrt{14a}}{2 - \sqrt{3b}}$ are radical expressions.

293. Some Properties of Radicals. A few important properties of radicals are given here. The fuller treatment is con-
-d in the chapter on Exponents.

294. I. $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$.

For example, $\sqrt{2}\cdot\sqrt{3}=\sqrt{6}$.

That this is true may be seen by squaring both members.

Thus, $(\sqrt{2}\cdot\sqrt{3})(\sqrt{2}\cdot\sqrt{3})=\sqrt{6}\cdot\sqrt{6}$,

or, $\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{3}=\sqrt{6}\cdot\sqrt{6}$,

or, $2\cdot 3=6$, which is known to be true.

In the same way, it may be seen that for every a and b, $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$.

In words:

The product of two square roots is the square root of the product of the numbers.

WRITTEN EXERCISES

Show by squaring that:

1. $\sqrt{3}\cdot\sqrt{5}=\sqrt{15}$.
2. $\sqrt{4}\cdot\sqrt{7}=\sqrt{28}$.
3. $\sqrt{3}\cdot\sqrt{7}=\sqrt{21}$.
4. $\sqrt{5}\cdot\sqrt{11}=\sqrt{55}$.
5. $\sqrt{2a}\cdot\sqrt{3b}=\sqrt{6ab}$.
6. $\sqrt{x^3}\cdot\sqrt{5y}=\sqrt{5x^3y}$.
7. $\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{5}=\sqrt{30}$.
8. $\sqrt{a}\cdot\sqrt{b}\cdot\sqrt{c}=\sqrt{abc}$.

295. II. $\sqrt{a^2b}=\sqrt{a^2}\ \sqrt{b}=a\sqrt{b}$.

In words:

Square factors may be taken from under the radical sign.

Thus, $\sqrt{18}=\sqrt{9}\cdot\sqrt{2}=\sqrt{3^2}\cdot\sqrt{2}=3\sqrt{2}$.

WRITTEN EXERCISES

Take all square factors from under the radical sign:

1. $\sqrt{20}$.
2. $\sqrt{27}$.
3. $\sqrt{50}$.
4. $\sqrt{48}$.
5. $\sqrt{45}$.
6. $\sqrt{75}$.
7. $\sqrt{24}$.
8. $\sqrt{32}$.
9. $\sqrt{12}$.
10. $\sqrt{40}$.
11. $\sqrt{500}$.
12. $\sqrt{128}$.
13. $\sqrt{8a^2}$.
14. $\sqrt{x^3}$.
15. $\sqrt{48x^3y^2}$.
16. $\sqrt{45a^7y^4}$.

296. III. $a\sqrt{b}=\sqrt{a^2}\cdot\sqrt{b}=\sqrt{a^2b}.$

In words:

Any factor outside the radical sign may be placed under the radical sign provided the factor is squared.

For example:

$$3\sqrt{2}=\sqrt{9}\cdot\sqrt{2}=\sqrt{18}.$$

WRITTEN EXERCISES

Place under one radical sign:

1. $6\sqrt{2}.$
2. $5\sqrt{3}.$
3. $2\cdot\sqrt{3}.$
4. $5\cdot\sqrt{7}.$
5. $3\cdot\sqrt{7}\cdot 2.$
6. $5\cdot\sqrt{3}\cdot\sqrt{2}.$
7. $2\cdot\sqrt{3}\cdot\sqrt{11}.$
8. $5\cdot\sqrt{3}\cdot\sqrt{7}.$
9. $4\sqrt{2}\cdot 3.$
10. $b\sqrt{2}.$
11. $2x\sqrt{3x}.$
12. $ab\sqrt{bc}.$
13. $t\sqrt{g}.$
14. $r\sqrt{\pi r}.$
15. $\frac{x}{3}\sqrt{18xy}.$
16. $a\sqrt{b-a}.$

PROCESSES

297. PREPARATORY.

Read and supply the blanks:

1. $3a+2a=(\quad)a.$ Similarly, $3\sqrt{2}+2\sqrt{2}=(\quad)\sqrt{2}.$
2. $5a-3a=(\quad)a.$ Similarly, $5\sqrt{2}-3\sqrt{2}=(\quad)\sqrt{2}.$
3. $7\sqrt{3}+3\sqrt{3}=(\quad)\sqrt{3}.$
4. $8\sqrt{5}-6\sqrt{5}=(\quad)\sqrt{5}.$
5. $\sqrt{75}-\sqrt{12}=5\sqrt{3}-2\sqrt{3}=(\quad)\sqrt{3}.$

298. Addition and Subtraction of Radical Expressions. Radicals can be united by addition or subtraction only when the same root is indicated and the expressions under the radical sign are the same in each.

ıen the expression cannot be put into this form the sum › difference can only be indicated.

299. *To add or subtract radical expressions having the same radical part, add or subtract the coefficients of their radical parts.*

For example:

$$2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}.$$

$$2\sqrt{12} + \sqrt{300} = 4\sqrt{3} + 10\sqrt{3} = 14\sqrt{3}.$$

Add $\sqrt{2}$, $-\sqrt{8}$, $\sqrt[3]{16}$, $\sqrt[3]{-54}$:

$$-\sqrt{8} = -2\sqrt{2};\quad \sqrt[3]{16} = 2\sqrt[3]{2};\quad \sqrt[3]{-54} = -3\sqrt[3]{2}.$$

$$\therefore \sqrt{2} - \sqrt{8} + \sqrt[3]{16} + \sqrt[3]{-54} = \sqrt{2} - 2\sqrt{2} + 2\sqrt[3]{2} - 3\sqrt[3]{2}$$
$$= -\sqrt{2} - \sqrt[3]{2}.$$

If the numerical value of the sum or the difference is needed, it can be found approximately by methods given later.

WRITTEN EXERCISES

Find the sum:

1. $\sqrt{2}$, $\sqrt{8}$, $\sqrt{18}$.
2. $\sqrt{75}$, $-\sqrt{12}$, $-\sqrt{3}$.
3. $\sqrt{8}$, $\sqrt{5}$, $-\sqrt{18}$.
4. $\sqrt{128}$, $-2\sqrt{50}$, $\sqrt{72}$.
5. $\sqrt[3]{40}$, $-\sqrt[3]{320}$, $\sqrt[3]{135}$.
6. $8\sqrt{48}$, $-\frac{1}{2}\sqrt{12}$, $4\sqrt{27}$.
7. $\sqrt[3]{72}$, $-3\sqrt[3]{9}$, $6\sqrt[3]{243}$.
8. $\sqrt{6}$, $\sqrt{24}$, $\sqrt{63}$.
9. $\sqrt{108}$, $-\sqrt{12}$, $\sqrt{48}$.
10. $\sqrt{75}$, $\sqrt{48}$, $-\sqrt{27}$.
11. $\sqrt{80}$, $\sqrt{20}$, $-\sqrt{45}$.
12. $\sqrt{44}$, $-\sqrt{99}$, $\sqrt{121}$.
13. $5\sqrt{24}$, $-\sqrt{54}$, $3\sqrt{96}$.
14. $\sqrt[3]{27x^4}$, $-\sqrt[3]{64x^4}$, $\sqrt[3]{16x^4}$.

300. Multiplication of Radical Expressions containing Square Roots. In multiplying expressions containing indicated square roots, make use of the relation $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$.

EXAMPLES

1.
$$\begin{array}{r} 3 - 4\sqrt{5} \\ 6 + 2\sqrt{3} \\ \hline 18 - 24\sqrt{5} \qquad\qquad\qquad \\ 6\sqrt{3} - 8\sqrt{15}. \\ \hline 18 - 24\sqrt{5} + 6\sqrt{3} - 8\sqrt{15}. \end{array}$$

2.
$$\begin{array}{l} 2 - \sqrt{3} \\ 5 + 2\sqrt{3} \\ \hline 10 - 5\sqrt{3} \\ \qquad 4\sqrt{3} - 6 \\ \hline 10 - \sqrt{3} - 6 = 4 - \sqrt{3}. \end{array}$$

WRITTEN EXERCISES

Multiply:

1. $2+\sqrt{5}$ by $2-\sqrt{5}$.
2. $1+\sqrt{3}$ by $2+\sqrt{5}$.
3. $2+\sqrt{3}$ by $2+\sqrt{3}$.
4. $\sqrt{2}+\sqrt{3}$ by $1-\sqrt{3}$.
5. $\sqrt{3}-\sqrt{5}$ by $\sqrt{3}+\sqrt{5}$.
6. $\sqrt{5}-\sqrt{6}$ by $\sqrt{5}-\sqrt{6}$.
7. $4+\sqrt{5}$ by $\sqrt{10}$.
8. $3-\sqrt{15}$ by $2+\sqrt{5}$.
9. $1+\sqrt{2}$ by $1-\sqrt{8}$.
10. $2\sqrt{3}-3\sqrt{5}$ by $\sqrt{3}-\sqrt{5}$.
11. $\sqrt{14}+\sqrt{7}$ by $\sqrt{8}-\sqrt{21}$.
12. $\sqrt{5}-\sqrt{48}$ by $\sqrt{5}+\sqrt{12}$.

301. Division of Square Roots. The quotient of the square roots of two numbers is the square root of the quotient of the numbers. In symbols, $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$.

Thus, $\dfrac{\sqrt{5}}{\sqrt{6}}=\sqrt{\dfrac{5}{6}}$, because, multiplying each member by itself,

$$\frac{\sqrt{5}}{\sqrt{6}}\cdot\frac{\sqrt{5}}{\sqrt{6}}=\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{5}{6}},$$

or $$\frac{\sqrt{5}\cdot\sqrt{5}}{\sqrt{6}\cdot\sqrt{6}}=\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{5}{6}},$$

or $$\frac{\sqrt{5^2}}{\sqrt{6^2}}=\sqrt{\left(\frac{5}{6}\right)^2},$$

or $$\frac{5}{6}=\frac{5}{6}, \text{ which is evidently true.}$$

ORAL EXERCISES

Read each of the following as a fraction under one radical sign:

1. $\dfrac{\sqrt{3}}{\sqrt{5}}$.
2. [illegible]
3. $\dfrac{\sqrt{5}}{\sqrt{7}}$.
4. $\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{1}}{\sqrt{2}}$.
5. $\dfrac{1}{\sqrt{5}}$.
6. $\dfrac{\sqrt{2}}{\sqrt{8}}$.
7. $\dfrac{\sqrt{5}}{\sqrt{15}}$.
8. $\dfrac{\sqrt{10}}{\sqrt{20}}$.

302. Rationalizing the Denominator. Multiplying both numerator and denominator of a fraction by an expression that will make the denominator rational is called **rationalizing the denominator.**

Thus, multiplying both numerator and denominator of $\frac{\sqrt{3}}{\sqrt{2}}$ by $\sqrt{2}$, we obtain

$$\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{2}\cdot\sqrt{2}} = \frac{\sqrt{6}}{2}.$$

WRITTEN EXERCISES

Rationalize the denominator of:

1. $\frac{1}{\sqrt{2}}$.
2. $\frac{1}{\sqrt{3}}$.
3. $\frac{1}{\sqrt{5}}$.
4. $\frac{2}{\sqrt{3}}$.
5. $\frac{3}{\sqrt{7}}$.
6. $\frac{\sqrt{2}}{\sqrt{3}}$.
7. $\frac{\sqrt{5}}{\sqrt{7}}$.
8. $\frac{5}{\sqrt{3}}$.
9. $\frac{\sqrt{5}}{\sqrt{3}}$.
10. $\frac{10}{\sqrt{5}}$.
11. $\frac{6}{\sqrt{3}}$.
12. $\frac{8}{\sqrt{2}}$.

303. Rationalizing Factors. When the denominator is of the form $\sqrt{a}+\sqrt{b}$ or $a+\sqrt{b}$, the rationalizing factor is the same binomial with the connecting sign changed, often called the **conjugate binomial.**

It is not necessary in elementary algebra to take up the rationalizing of more complicated denominators.

EXAMPLES

1. Rationalize the denominator in $\frac{3}{2-\sqrt{5}}$.

The conjugate of $2-\sqrt{5}$ is $2+\sqrt{5}$.

Then, $\frac{3}{2-\sqrt{5}} = \frac{3(2+\sqrt{5})}{(2-\sqrt{5})(2+\sqrt{5})} = \frac{6+3\sqrt{5}}{4-5} = -(6+3\sqrt{5})$.

2. Rationalize the denominator in $\dfrac{2+\sqrt{3}}{\sqrt{3}+\sqrt{5}}$.

The conjugate of $\sqrt{3}+\sqrt{5}$ is $\sqrt{3}-\sqrt{5}$.

Then, $$\frac{2+\sqrt{3}}{\sqrt{3}+\sqrt{5}}=\frac{(2+\sqrt{3})(\sqrt{3}-\sqrt{5})}{(\sqrt{3}+\sqrt{5})(\sqrt{3}-\sqrt{5})}=\frac{2\sqrt{3}+3-2\sqrt{5}-\sqrt{15}}{3-5}$$

$$=-\frac{3+2\sqrt{3}-2\sqrt{5}-\sqrt{15}}{2}.$$

WRITTEN EXERCISES

Rationalize the denominators:

1. $\dfrac{2+\sqrt{3}}{3+\sqrt{3}}$.
2. $\dfrac{3\sqrt{3}+2\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.
3. $\dfrac{1}{1-\sqrt{2}}$.
4. $\dfrac{3}{2-\sqrt{5}}$.
5. $\dfrac{5}{\sqrt{3}+\sqrt{7}}$.
6. $\dfrac{2-\sqrt{5}}{3-\sqrt{5}}$.
7. $\dfrac{3}{\sqrt{5}+\sqrt{2}}$.
8. $\dfrac{\sqrt{5}+2\sqrt{2}}{4-2\sqrt{2}}$.
9. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.
10. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.
11. $\dfrac{2+4\sqrt{7}}{2\sqrt{7}-1}$.
12. $\dfrac{2\sqrt{15}-6}{\sqrt{5}+2\sqrt{2}}$.

RADICAL EQUATIONS

304. *To solve equations in which only a single square root occurs, transpose so that the square root constitutes one member. Square both members and solve the resulting equation.*

EXAMPLE

Solve: $$2x-3=\sqrt{x^2+6x-6}. \quad (1)$$

Squaring both members, $$4x^2-12x+9=x^2+6x-6. \quad (2)$$

$$3x^2-18x+15=0. \quad (3)$$

... by 3, $$x^2-6x+5=0. \quad (4)$$

$$x=5 \text{ or } 1. \quad (5)$$

... appears that 5 satisfies the given equation, taking the ... 1 satisfies the equation $2x-3=-\sqrt{x^2+6x-6}$.

1. It must be remembered that the equation resulting from squaring will usually not be *equivalent* to the given equation (Sec. 232, p. 186). It may have additional roots, and trial must determine which of the roots found satisfy the given equation.

2. In order that the given problem may be definite, the radical must be taken with a given sign. If every possible square root is meant, two different equations are really given for solution. Thus, unless restricted, $2x = \sqrt{4 - 3x}$ is a compact way of uniting the two different equations, $2x = +\sqrt{4 - 3x}$, and $2x = -\sqrt{4 - 3x}$. If solved as indicated above, it appears that the first is satisfied when $x = \frac{1}{2}$, the second when $x = -\frac{4}{3}$.

3. In the exercises of the following set the radical sign is to be understood to mean the *positive* square root.

WRITTEN EXERCISES

Solve:

1. $x = \sqrt{10x + 7}$.
2. $x = \sqrt{b + x - bx}$.
3. $3x - 7\sqrt{x} = -2$.
4. $\sqrt{x + 5} - x = -1$.
5. $\dfrac{\sqrt{x} - 1}{3} = \dfrac{x}{16}$.
6. $x + 5\sqrt{37 - x} = 43$.
7. $30 = x - 29\sqrt{x}$.
8. $x = 2 + \sqrt{\dfrac{3 - 11x}{6}}$.
9. $x - \sqrt{x + 2} = 3$.
10. $2x + 1 = \sqrt{x - 4}$.
11. $\sqrt{100 - x^2} = 10 - x$.
12. $x + \sqrt{2s - x^2} = 6$.

SUMMARY

I. **Definitions.**

1. Rational numbers are integers and other numbers expressible as the quotient of two integers. Sec. 288.

2. An *irrational number* is any number not rational. Sec. 289.

3. A *radical* is an indicated root of a number. Sec. 290.

4. A *surd* is an irrational number that is an indicated root of a rational *number*. Sec. 291.

5. A *radical expression* is an expression involving one or more radicals. Sec. 292.

II. **Properties and Operations.**

1. The product of two square roots is the square root of the product of the numbers. Sec. 294.

2. Square factors may be taken from under the radical sign. Sec. 295.

3. Any factor outside the radical sign may be placed under the radical sign provided the factor is squared. Sec. 296.

4. Radical expressions whose radical parts are the same may be added or subtracted by adding or subtracting the coefficients of their radical parts. Sec. 299.

5. The quotient of two square roots is the square root of the quotient of the numbers. Sec. 301.

6. Multiplying both numerator and denominator of a fraction by a factor that will make the denominator rational is called *rationalizing the denominator.* Sec. 302.

7. Radical equations involving only a single square root are solved by transposing so that the square root constitutes one member, squaring and solving the resulting equation. Sec. 304.

REVIEW

WRITTEN EXERCISES

Simplify:

1. $\dfrac{\sqrt{5}}{\sqrt{60}}$.
2. $\sqrt{\frac{27}{8}} - \sqrt{\frac{3}{8}}$.
3. $\sqrt{6} \cdot \sqrt{125}$.
4. $\dfrac{\sqrt{10}}{\sqrt{40}}$.
5. $\sqrt{50} + \sqrt{128}$.
6. $\sqrt{3} \div \sqrt{5}$.
7. $\dfrac{3\sqrt{3} + 2\sqrt{2}}{\sqrt{3} - \sqrt{2}}$.
8. $\sqrt{6} \div \sqrt{2}$.
9. $(2 + \sqrt{3})^2$.
10. $(5 + \sqrt{7})(5 - \sqrt{7})$.
11. $(2\sqrt{3} + 3\sqrt{5}) \div \sqrt{15}$.
12. $(\sqrt{6} + \sqrt{15})(\sqrt{8} - \sqrt{20})$.

Solve:

13. $\sqrt{x+5} = x - 7$.
14. $\sqrt{2x+7} = \frac{x}{3}$.
15. $\dfrac{x-10}{2} + \dfrac{x-1}{3} + \dfrac{\sqrt{2x-1}}{2} = 0$.
16. $x + \sqrt{x+5} = 2x - 1$.
17. $x - 2 + \sqrt{2 - x} = 0$.
18. $x + \sqrt{9 - x^2} = 6$.

SUPPLEMENTARY WORK

ADDITIONAL EXERCISES

Express with rational denominators, and with at most one radical sign in the dividend:

1. $\sqrt{12} \div \sqrt{3}$.
2. $\sqrt{7} \div \sqrt{11}$.
3. $2\sqrt{24} \div 2\sqrt{6}$.
4. $2 \div 3\sqrt{5}$.
5. $\dfrac{4}{\sqrt{5}-1}$.
6. $1 \div (\sqrt{2}-10)$.
7. $\sqrt{2} \div (\sqrt{2}-\sqrt{3})$.
8. $(2\sqrt{6}+5\sqrt{12}) \div \sqrt{6}$.
9. $(5\sqrt{18}-8\sqrt{50}) \div 2\sqrt{2}$.
10. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}$.
11. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.
12. $\dfrac{\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.
13. $\dfrac{2+4\sqrt{7}}{2\sqrt{7}-1}$.
14. $\dfrac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}$.

Square Root of Binomials of the Form $a+\sqrt{b}$

Binomials of the form $a+\sqrt{b}$ can often be put into the form $x+y+2\sqrt{xy}$, or $(\sqrt{x}+\sqrt{y})^2$, and hence the square root, $\sqrt{x}+\sqrt{y}$, of the binomial can be written at once.

EXAMPLES

1. Find the square root of $4+2\sqrt{3}$.

$$4+2\sqrt{3}=3+1+2\sqrt{3\cdot 1}.$$

Hence, $x+y=3+1$ and $xy=3\cdot 1$, from which $x=3$ and $y=1$.

$$\therefore \sqrt{4+2\sqrt{3}}=\pm(\sqrt{3}+\sqrt{1})=\pm(\sqrt{3}+1).$$

The coefficient of the radical must be made 2 in order to apply the formula $x+y+2\sqrt{xy}$.

2. Find the square root of $3-\sqrt{8}$.

$$3-\sqrt{8}=3-\sqrt{4\cdot 2}=3-2\sqrt{2}.$$

$$\therefore x+y=3, \text{ and } xy=2. \quad \therefore x=2,\ y=1 \text{ by inspection.}$$

$$\therefore \sqrt{3-\sqrt{8}}=\pm(\sqrt{2}-\sqrt{1})=\pm(\sqrt{2}-1).$$

3. Find the square root of $7+4\sqrt{3}$.

$$7+4\sqrt{3}=7+2\sqrt{4\cdot 3};\ x+y=7,\ xy=12;\ \therefore x=4,\ y=3.$$

$$\therefore \sqrt{7+4\sqrt{3}}=\pm(\sqrt{4}+\sqrt{3})=\pm(2+\sqrt{3}).$$

The square root as a whole may be taken positively or negatively, as in the case of rational roots.

The solution of these problems depends upon finding two numbers whose sum and product are given. This can sometimes be done by inspection, but the general problem is one of simultaneous equations. See Sec. 394, p. 372.

WRITTEN EXERCISES

Find the square root of:

1. $11+6\sqrt{2}$.
2. $8-2\sqrt{15}$.
3. $49-12\sqrt{10}$.
4. $41-24\sqrt{2}$.
5. $2\frac{1}{4}-\sqrt{5}$.
6. $2\frac{1}{3}-\frac{4}{3}\sqrt{3}$.
7. $17+12\sqrt{2}$.
8. $\frac{3}{2}\sqrt{5}+3\frac{1}{2}$.
9. $56-24\sqrt{5}$.

CHAPTER XVIII

EXPONENTS

LAWS OF EXPONENTS

305. PREPARATORY.

1. What is the meaning of a^2? Of a^4? Of a^n? (Sec. 32, p. 17.)

2. What is the meaning of $\sqrt{a^2}$? Of $\sqrt[3]{a^3}$? Of $\sqrt[10]{a^{10}}$? Of $\sqrt[n]{a^n}$?

3. $a^2 \cdot a^3 = ?$ $a^3 \cdot a^2 = ?$ $a^5 \cdot a^3 = ?$ $a^8 \cdot a^5 = ?$ (Sec. 99, p. 63.)

4. $a^3 \div a^2 = ?$ $a^4 \div a^2 = ?$ $a^5 \div a^2 = ?$ $b^{10} \div b^4 = ?$

5. $(a^2)^2 = ?$ $(a^2)^3 = ?$ $(c^5)^2 = ?$ $(x^{10})^3 = ?$

306. We shall soon define negative and fractional exponents, but until this is done literal exponents are to be understood to represent positive integers.

307. Law of Exponents in Multiplication.

I. $$a^m \cdot a^r = a^{m+r}.$$

For $\quad a^m = a \cdot a \cdot a \cdots$ to m factors,

and $\quad a^r = a \cdot a \cdot a \cdots$ to r factors.

$$\therefore a^m \cdot a^r = (a \cdot a \cdot a \cdots \text{ to } m \text{ factors})(a \cdot a \cdot a \cdots \text{ to } r \text{ factors})$$
$$= a \cdot a \cdot a \cdot a \cdots \text{ to } m + r \text{ factors}$$
$$= a^{m+r}, \text{ by the definition of exponent.}$$

Similarly, $\quad a^m \cdot a^r \cdot a^p \cdots = a^{m+r+p} \cdots.$

ORAL EXERCISES

Multiply:

1. $a^2 \cdot a^4$.
2. $a^3 \cdot a^3$.
3. $a^5 \cdot a^7$.
4. $m^1 \cdot m^5$.
5. $x \cdot x$.
6. $2^3 \cdot 2^4$.
7. $(-1)^3 \cdot (-1)^5$.
8. $6^2 \cdot 6^2$.
9. $5 \cdot 5 \cdot 5^2$.
10. $2^3 \cdot 2^3 \cdot 2^2$.
11. $7 \cdot 7^2 \cdot 7^3$.
12. $3 \cdot 3^5 \cdot 3^2$.
13. $(-1)^2 \cdot (-1)^3 \cdot (-1)^5$.
14. $(-a)^2 \cdot (-a)^4 \cdot (-a)$.

308. Law of Exponents in Division.

II. $$\frac{a^m}{a^r} = a^{m-r}, \text{ if } m > r.$$

For $a^m = a \cdot a \cdot a \cdots m$ factors,

and $a^r = a \cdot a \cdot a \cdots r$ factors.

$\therefore \dfrac{a^m}{a^r} = \dfrac{a \cdot a \cdots m \text{ factors}}{a \cdot a \cdots r \text{ factors}}$

$= a \cdot a \cdot a \cdots m - r$ factors, canceling the r factors from both terms

$= a^{m-r}$, by definition of exponent.

ORAL EXERCISES

Divide:

1. $\dfrac{a^4}{a^2}$.
2. $\dfrac{a^5}{a^3}$.
3. $\dfrac{a^7}{a^5}$.
4. $\dfrac{2^4}{2^2}$.
5. $\dfrac{a^8}{a^5}$.
6. $\dfrac{(-a)^5}{(-a)}$.
7. $\dfrac{(-1)^5}{(-1)^3}$.
8. $\dfrac{(ab)^2}{(ab)}$.
9. $\dfrac{6^3}{6^2}$.
10. $\dfrac{5^4}{5^3}$.
11. $\dfrac{4\pi r^2}{4r}$.
12. $\dfrac{x^5}{x}$.
13. $\dfrac{x^2y}{x}$.
14. $\dfrac{mv^2}{v}$.
15. $\dfrac{\frac{1}{2}gt^2}{t}$.
16. $\dfrac{\pi r^3}{r}$.

309. Laws of Exponents for Powers.

III. $$(a^m)^r = a^{mr}.$$

For $(a^m)^r = a^m \cdot a^m \cdot a^m \cdots$ to r factors

$= (a \cdot a \cdots$ to m factors$)(a \cdot a \cdots$ to m factors$)$ to r such parentheses

$= a \cdot a \cdot a \cdots$ to mr factors

$= a^{mr}$, by definition of exponent.

ORAL EXERCISES

Apply this law to:

1. $(4^3)^2$.
2. $(3^2)^5$.
3. $(2^5)^4$.
4. $(a^3)^2$.
5. $(a^2)^3$.
6. $(a^5)^2$.
7. $(x^2)^5$.
8. $(x^3)^3$.
9. $(y^4)^4$.
10. $[(b)^4]^5$.
11. $[(-a)^5]^2$.
12. $[(-8)^2]^3$.

IV. $$(ab)^n = a^n b^n.$$

For $(ab)^n = (ab)\cdot(ab)\ \cdots$ to n factors

$= (a\cdot a \cdots \text{to } n \text{ factors})(b \cdot b \cdots \text{to } n \text{ factors})$

$= a^n b^n$, by definition of exponent.

Similarly, $(abc\cdots)^n = a^n b^n c^n \cdots$.

ORAL EXERCISES

Apply this law to:

1. $(8\cdot 3)^2$.	4. $(ab)^5$.	7. $(mn)^p$.	10. $(x^2y)^3$.
2. $(4\cdot 5)^2$.	5. $(cd)^3$.	8. $(xy)^a$.	11. $(x^a y^2)^5$.
3. $(2\cdot 5)^3$.	6. $(abc)^4$.	9. $(ab)^{2r}$.	12. $(ax^3)^b$.

V. $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

For $\left(\frac{a}{b}\right)^n = \frac{a}{b}\cdot\frac{a}{b}\cdot\cdot\cdot$ to n factors

$= \dfrac{a\cdot a \cdots \text{to } n \text{ factors}}{b\cdot b \cdots \text{to } n \text{ factors}}$

$= \dfrac{a^n}{b^n}$, by definition of exponent.

ORAL EXERCISES

Apply this law to:

1. $(\frac{2}{3})^2$.
2. $(\frac{1}{5})^3$.
3. $(\frac{2}{5})^5$.
4. $(\frac{3}{5})^4$.
5. $(-\frac{6}{7})^2$.
6. $(-\frac{3}{4})^3$.
7. $\left(\frac{a}{b}\right)^2$.
8. $\left(\frac{a}{b}\right)^5$.
9. $\left(\frac{c}{d}\right)^3$.
10. $\left(\frac{x}{y}\right)^n$.
11. $\left(\frac{m}{n}\right)^a$.
12. $\left(\frac{p}{q}\right)^x$.
13. $\left(\frac{2a}{b}\right)^5$.
14. $\left(\frac{c}{5d}\right)^7$.
15. $\left(\frac{-xy}{z}\right)^5$.

310. Collected Laws of Exponents.

I.	$a^m \cdot a^r = a^{m+r}$.		Sec. 307.
II.	$a^m \div a^r = a^{m-r}$.	$(m > r.)$	Sec. 308.
III.	$(a^m)^r = a^{mr}$.		Sec. 309.
IV.	$(ab)^n = a^n b^n$.		Sec. 309.
V.	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.		Sec. 309.

FRACTIONAL EXPONENTS

311. Hitherto we have spoken only of positive integers as exponents, the exponent meaning the number of times the base is used as a factor. This meaning does not apply to fractional and negative exponents, because it does not mean anything to speak of using a as a factor $\frac{2}{3}$ of a time, or -6 times. But it is possible to find meanings for fractional and negative exponents such that they will conform to the laws of integral exponents.

312. Preparatory.

Find the meaning of $a^{\frac{1}{2}}$.

Assuming that Law I applies, $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a$,

or, $(a^{\frac{1}{2}})^2 = a$.

That is, $a^{\frac{1}{2}}$ is one of the two equal factors of a,

or, $a^{\frac{1}{2}} = \sqrt{a}$.

Thus, the fractional exponent $\frac{1}{2}$ means *square root*.

Similarly, $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$,

or, $(a^{\frac{1}{3}})^3 = a$.

That is, $a^{\frac{1}{3}}$ is one of the three equal factors of a,

or, $a^{\frac{1}{3}} = \sqrt[3]{a}$.

Thus, the fractional exponent $\frac{1}{3}$ means *cube root*.

WRITTEN EXERCISES

Find similarly the meaning of:

1. $a^{\frac{1}{4}}$.	3. $x^{\frac{1}{3}}$.	5. $n^{\frac{1}{6}}$.	7. $m^{\frac{1}{8}}$.
2. $a^{\frac{1}{5}}$.	4. $b^{\frac{1}{10}}$.	6. $c^{\frac{1}{7}}$.	8. $a^{\frac{1}{20}}$.

313. The meaning of $a^{\frac{1}{n}}$ is found as follows:

Assuming that Law I applies,

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdots \text{ to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\cdots \text{ to } n \text{ terms}}$$
$$= a^{n \cdot \frac{1}{n}}$$
$$= a^1 = a.$$

That is, $a^{\frac{1}{n}}$ is one of n equal factors of a, or $a^{\frac{1}{n}} = \sqrt[n]{a}$.

314. Preparatory.

Find the meaning of $a^{\frac{2}{3}}$.

Assuming that Law I applies, $a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2$, or $(a^{\frac{2}{3}})^3 = a^2$.

That is, $a^{\frac{2}{3}}$ is one of the three equal factors of a^2, or $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

315. The meaning of $a^{\frac{p}{q}}$ is found as follows:

Assuming that Law I applies,

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \cdots \text{ to } q \text{ factors} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}\cdots \text{ to } q \text{ terms}}$$
$$= a^{q \cdot \frac{p}{q}} = a^p.$$

That is, $a^{\frac{p}{q}}$ is one of the q equal factors of a^p,

or, $$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Similarly, $$a^{\frac{p}{q}} = a^{\frac{1}{q} \cdot p} = (\sqrt[q]{a})^p.$$

In words:

a *with the exponent* $\frac{\text{p}}{\text{q}}$ *denotes the* q*th root of the* p*th power of* a, *or the* p*th power of the* q*th root of* a.

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This definition applies when p and q are positive integers. The meaning of negative fractional exponents is found in Sec. 324, p. 286.

ORAL EXERCISES

1. State the meaning of:

1. $5^{\frac{2}{3}}$.	3. $6^{\frac{1}{2}}$.	5. $4^{\frac{5}{7}}$.	7. $8^{\frac{3}{4}}$.
2. $a^{\frac{3}{4}}$.	4. $a^{\frac{2}{5}}$.	6. $b^{\frac{5}{6}}$.	8. $c^{\frac{4}{3}}$.

Find the value of:

9. $8^{\frac{4}{3}}$.	10. $16^{\frac{3}{4}}$.	11. $25^{\frac{3}{2}}$.	12. $32^{\frac{3}{5}}$.

WRITTEN EXERCISES

Express with fractional exponents:

1. $\sqrt[4]{a^3}$.	11. $\sqrt[5]{\frac{32\,a^5b^8}{c^{15}}}$.	19. $\sqrt[n]{a}$.
2. $\sqrt[5]{a^2}$.	12. $\sqrt[3]{\frac{16\,a^2}{9\,b^2}}$.	20. $\sqrt[3]{(a-b)^2}$.
3. $\sqrt[3]{mn}$.	13. $\sqrt[3]{\frac{-8\,x^3y^6}{5}}$.	21. $\sqrt[n]{a}\cdot\sqrt[m]{b}$.
4. $\sqrt{a}\sqrt[3]{b}$.	14. $\sqrt[6]{\frac{64\,x^{12}}{y}}$.	22. $\sqrt[n]{a^m}$.
5. $\sqrt[3]{ab}$.	15. $\sqrt{a+b}$.	23. $\sqrt[m]{a^n}$.
6. $\sqrt[3]{ab^2xy}$.	16. $\sqrt{a^3+b^3}$.	24. $\sqrt[pn]{b^{2n}}$.
7. $\sqrt[3]{\frac{x^4y^2}{4}}$.	17. $\sqrt[3]{b^2}\cdot\sqrt{a}$.	25. $\sqrt[2m]{a^{mn}}$.
8. $\sqrt{\frac{9\,a^2b}{x}}$.	18. $\sqrt[5]{-a}\cdot\sqrt[5]{-b}$.	26. $\sqrt[2mn]{a^{2mn}}$.
9. $\sqrt[8]{16\,x^4y^8}$.		27. $\sqrt[mn]{-a^nb^m}$.
10. $\sqrt{m}\sqrt[3]{n^2}\sqrt[5]{p}$.		28. $\sqrt[n]{a}\cdot\sqrt[p]{q}$.
		29. $\sqrt[2n]{abc}$.
		30. $\sqrt[p]{a^q}\cdot\sqrt[q]{a^p}$.

316. The definition of positive fractional exponents has been found as a consequence of the assumption that Law I applies to them. It can be shown that the other laws of Sec. 310, p. 278, also apply to this class of exponents, as thus defined, and we shall so apply them, although the proof is omitted here.

317. According to Law I (Sec. 307), *when the bases are the same, the exponent of the product is found by adding the exponents.*

For example: $a^{\frac{1}{2}} \cdot a^{\frac{3}{4}} = a^{\frac{1}{2}+\frac{3}{4}} = a^{\frac{5}{4}}.$

A general formula for this statement is,

$$a^{\frac{m}{n}} \cdot a^{\frac{p}{q}} = a^{\frac{mq}{nq}+\frac{np}{nq}} = a^{\frac{mq+np}{nq}}.$$

The number a, or the base, must be the same in all factors. When it is not, as in $a^{\frac{1}{2}} \cdot b^{\frac{3}{4}}$, the product cannot be found by adding the exponents.

WRITTEN EXERCISES

Find the products:

1. $a^{\frac{1}{2}} \cdot a^{\frac{2}{3}}.$
2. $4^2 \cdot 4^{\frac{3}{5}}.$
3. $7^{\frac{1}{2}} \cdot 7^{\frac{3}{5}}.$
4. $a^{\frac{3}{2}} \cdot a^{\frac{2}{3}}.$
5. $b^{\frac{1}{2}} \cdot b^{\frac{1}{3}}.$
6. $a^{\frac{2}{3}} \cdot a^{\frac{3}{4}}.$
7. $m^{\frac{3}{4}} \cdot m^{\frac{3}{5}}.$
8. $a^{\frac{5}{6}} \cdot a^{\frac{1}{2}}.$
9. $r^{\frac{1}{n}} \cdot r^{\frac{1}{n}}.$
10. $a^{\frac{1}{n}} \cdot a^{\frac{1}{m}}.$
11. $x^{\frac{1}{a}} \cdot x^{\frac{1}{b}}.$
12. $x^{\frac{m}{n}} \cdot x^{\frac{p}{n}}.$
13. $p^{\frac{2}{3}} \cdot p^{\frac{1}{2}} \cdot p.$
14. $a^{\frac{4}{5}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{1}{2}}.$

318. According to Law II (Sec. 310), *when the bases are the same, the exponent of the quotient is found by taking the difference between the exponents.*

For example: $a^{\frac{2}{3}} \div a^{\frac{1}{2}} = a^{\frac{2}{3}-\frac{1}{2}} = a^{\frac{1}{6}}.$

$$a \div a^{\frac{3}{4}} = a^{1-\frac{3}{4}} = a^{\frac{1}{4}}.$$

A general formula for this statement is,

$$a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}} = a^{\frac{mq-np}{nq}}.$$

$\frac{m}{n}$ is here supposed to be greater than $\frac{p}{q}$, but this restriction will be removed later.

WRITTEN EXERCISES

Find the quotients:

1. $a^{\frac{3}{4}} \div a^{\frac{1}{2}}$.
2. $a \div a^{\frac{7}{12}}$.
3. $a \div a^{\frac{1}{n}}$.
4. $a \div a^{\frac{p}{q}}$.
5. $a^{\frac{4}{5}} \div a^{\frac{3}{4}}$.
6. $x^{\frac{7}{8}} \div x^{\frac{3}{4}}$.
7. $a^{\frac{3}{4}} \div a^{\frac{2}{3}}$.
8. $x^{\frac{2}{5}} \div x^{\frac{1}{7}}$.
9. $ab \div (ab)^{\frac{1}{7}}$.
10. $\left(\frac{1}{a}\right)^{\frac{5}{6}} \div \left(\frac{1}{a}\right)^{\frac{1}{2}}$.
11. $5^{\frac{2}{3}} \div 5^{\frac{1}{2}}$.
12. $x^{\frac{m}{n}} \div x^{\frac{p}{n}}$
13. $m^{\frac{1}{a}} \div m^{\frac{1}{b}}$.
14. $6^{\frac{2}{3}} \div 6^{\frac{1}{6}}$.
15. $m^{\frac{3}{5}} \div m^{\frac{1}{5}}$.
16. $p^{\frac{4}{5}} \div p^{\frac{3}{4}}$.

319. According to Law III (Sec. 310), *when an exponent is applied to a base having an exponent, the product of the exponents is the exponent of the result.*

For example:

$$(a^{\frac{1}{2}})^2 = a^{2 \cdot \frac{1}{2}} = a^1 = a.$$

$$(a^2)^{\frac{1}{4}} = a^{2 \cdot \frac{1}{4}} = a^{\frac{2}{4}} = a^{\frac{1}{2}}.$$

$$(a^{\frac{1}{3}})^{\frac{2}{5}} = a^{\frac{1}{3} \cdot \frac{2}{5}} = a^{\frac{2}{15}}.$$

A general formula for this statement is,

$$(a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{m}{n} \cdot \frac{p}{q}} = a^{\frac{mp}{nq}}.$$

ORAL EXERCISES

Simplify:

1. $(2^{\frac{1}{2}})^{\frac{1}{3}}$.
2. $(3^{\frac{1}{3}})^2$.
3. $(3^{\frac{2}{3}})^3$.
4. $(5^{\frac{3}{4}})^2$.
5. $(b^{\frac{1}{3}})^{\frac{1}{2}}$.
6. $(a^{\frac{1}{2}})^{\frac{1}{3}}$.
7. $(a^{\frac{2}{3}})^{\frac{5}{6}}$.
8. $(a^{\frac{3}{4}})^{\frac{2}{3}}$.
9. $(x^{\frac{5}{6}})^{\frac{7}{8}}$.
10. $(y^9)^{\frac{1}{3}}$.
11. $(5^{\frac{1}{2}})^{\frac{2}{3}}$.
12. $(3^{\frac{2}{3}})^{\frac{1}{3}}$.
13. $(a^{\frac{1}{5}})^{\frac{1}{2}}$.
14. $(a^{\frac{3}{4}})^4$.
15. $(10^{\frac{1}{6}})^{\frac{1}{3}}$.
16. $[(a+b)^{\frac{1}{2}}]^{\frac{1}{2}}$.
17. $[(a-b)^{\frac{1}{3}}]^{\frac{3}{4}}$.
18. $[(a^2-b^2)^{\frac{1}{4}}]^{\frac{1}{5}}$.
19. $[(x^n-y^n)^{\frac{1}{p}}]^{\frac{p}{q}}$.

320. According to Law IV (Sec. 310), *an exponent affecting a product is applied to each factor,* and according to Law V (Sec. 310), *an exponent affecting a fraction is applied to both numerator and denominator.*

For example:

$$(ab)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}.$$

$$(ab^{\frac{1}{2}}c^2)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{4}}c.$$

$$(8\,x^6y^2z)^{\frac{1}{3}} = 8^{\frac{1}{3}}x^{\frac{6}{3}}y^{\frac{2}{3}}z^{\frac{1}{3}} = 2\,x^2y^{\frac{2}{3}}z^{\frac{1}{3}}.$$

$$(a^m b^{\frac{m}{n}} c)^{\frac{p}{q}} = a^{\frac{mp}{q}} \cdot b^{\frac{mp}{nq}} \cdot c^{\frac{p}{q}}.$$

$$\left(\frac{x^2}{y^8}\right)^{\frac{1}{4}} = \frac{(x^2)^{\frac{1}{4}}}{(y^8)^{\frac{1}{4}}} = \frac{x^{\frac{2}{4}}}{y^{\frac{8}{4}}} = \frac{x^{\frac{1}{2}}}{y^2}.$$

A general formula for this statement is,

$$\left(\frac{a^{\frac{m}{n}}}{b^{\frac{p}{q}}}\right)^{\frac{r}{s}} = a^{\frac{mr}{ns}} + b^{\frac{pr}{qs}}, \text{ or } \frac{a^{\frac{mr}{ns}}}{b^{\frac{pr}{qs}}}.$$

WRITTEN EXERCISES

Simplify:

1. $(a^3b^2)^{\frac{1}{2}}$.
2. $(a^2b^{\frac{1}{3}})^3$.
3. $(a^m b^n)^{\frac{1}{p}}$.
4. $(x^{\frac{1}{2}}y^{\frac{1}{4}})^8$.
5. $(a^{\frac{2}{3}}b^{\frac{1}{4}})^{\frac{1}{2}}$.
6. $(a^{\frac{3}{4}} \cdot b^{\frac{7}{8}})^8$.
7. $(86\,x^6y)^{\frac{1}{3}}$.
8. $\left(\frac{32\,a^5b^{10}}{c^{15}}\right)^{\frac{1}{5}}$.
9. $\left(\frac{64\,x^{12}}{y}\right)^{\frac{1}{6}}$.
10. $\left(\frac{9\,a^2b}{x}\right)^{\frac{1}{2}}$.
11. $\left(\frac{x^4y^3}{4}\right)^2$.
12. $(16\,x^4y^8)^{\frac{1}{4}}$.
13. $(a^6bn^{915})^{\frac{1}{3}}$.
14. $(27\,a^{12}b^9c^6)^{\frac{1}{3}}$.
15. $(m^{\frac{1}{2}}n^{\frac{1}{4}}p^{\frac{1}{5}})^{20}$.

321. When the bases are different and the fractional exponents are different, the exponents must have a common denominator, before any simplification by multiplication or division is possible.

For example : $a^{\frac{1}{2}}b^{\frac{1}{3}} = a^{\frac{3}{6}}b^{\frac{2}{6}} = (a^3b^2)^{\frac{1}{6}}.$

A general formula for this statement is,

$$a^{\frac{m}{n}}b^{\frac{p}{q}} = a^{\frac{mq}{nq}} \cdot b^{\frac{np}{nq}} = (a^{mq} \cdot b^{np})^{\frac{1}{nq}}.$$

This is called *simplifying by reducing exponents to the same order.*

WRITTEN EXERCISES

Simplify by reducing the exponents to the same order:

1. $a^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$.
2. $a^{\frac{2}{3}} \cdot b^{\frac{3}{4}}$.
3. $v^{\frac{1}{2}} \cdot b^{\frac{1}{3}}$.
4. $a^{\frac{1}{5}} \cdot b^{\frac{1}{4}}$.
5. $a^{\frac{1}{3}} \cdot b^{\frac{3}{4}}$.
6. $b^{\frac{1}{2}} \cdot 6^{\frac{1}{6}}$.
7. $5^{\frac{1}{5}} \cdot b^{\frac{1}{4}}$.
8. $a^{\frac{1}{2}} \div b^{\frac{1}{3}}$.
9. $m^{\frac{1}{3}} \cdot n^{\frac{1}{2}} \cdot p^{\frac{1}{4}}$.
10. $m^{\frac{2}{3}} \cdot n^{\frac{3}{5}} \div p^{\frac{1}{2}}$.
11. $p^{\frac{1}{n}} \cdot q^{\frac{1}{m}} \cdot r^{\frac{1}{3}}$.
12. $x^{\frac{p}{q}} \cdot y^{\frac{m}{n}} \cdot z^{\frac{1}{n}}$.

322. It is usually preferable to indicate roots by fractional exponents instead of by radical signs, since operations are thus more easily seen.

COMPARISON

	By Radicals	By Exponents
1.	$\sqrt{a}\sqrt[3]{a} = \sqrt[6]{a^3}\sqrt[6]{a^2} = \sqrt[6]{a^3a^2} = \sqrt[6]{a^5}$.	$a^{\frac{1}{2}}a^{\frac{1}{3}} = a^{\frac{3}{6}}a^{\frac{2}{6}} = a^{\frac{3}{6}+\frac{2}{6}} = a^{\frac{5}{6}}$.
2.	$\sqrt{b} \div \sqrt[3]{b} = \sqrt[6]{b^3} \div \sqrt[6]{b^2} = \sqrt[6]{b^3 \div b^2} = \sqrt[6]{b}$.	$b^{\frac{1}{2}} \div b^{\frac{1}{3}} = b^{\frac{3}{6}} \div b^{\frac{2}{6}} = b^{\frac{3}{6}-\frac{2}{6}} = b^{\frac{1}{6}}$.

WRITTEN EXERCISES

Simplify by use of fractional exponents as in the examples above:

1. $2^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}$.
2. $\sqrt{\frac{49}{121}}$.
3. $\sqrt{81\,a^{10}}$.
4. $\sqrt{5} \cdot \sqrt{75}$.
5. $3^{\frac{1}{2}} \cdot 2 \cdot 7^{\frac{1}{2}}$.
6. $3 \cdot 5^{\frac{1}{2}} \cdot 2\sqrt{3}$.

7. $\sqrt{8} \cdot 3\sqrt{2}$.

8. $2\sqrt{3} \cdot 3 \cdot \sqrt{10}$.

9. $2\sqrt[3]{3} \cdot 3\sqrt[3]{2}$.

10. $5^{\frac{1}{2}} \cdot 3 \cdot 5^{\frac{1}{3}}$.

11. $\sqrt{7} \cdot 11^{\frac{1}{2}}$.

12. $-8\sqrt{2} \cdot 12\sqrt{3}$.

13. $-\sqrt{12} \cdot 2\sqrt{3}$.

14. $a\sqrt{b} \cdot b\sqrt{a}$.

15. $(\sqrt{2}-\sqrt{3})2\sqrt{3}$.

16. $3^{\frac{1}{2}}(6^{\frac{1}{2}} - 2 \cdot 5^{\frac{1}{2}})$.

17. $(a\sqrt[3]{x})^2$.

18. $\sqrt{a^6b^8}$.

19. $\sqrt[3]{8a^5b^6}$.

20. $(27xy^2)^{\frac{1}{3}}$.

21. $(16x^4p)^{\frac{1}{4}}$.

22. $\sqrt[5]{32a^8b^{10}}$.

23. $\sqrt{36m^4n^2}$.

24. $\sqrt{64m^8n^6}$.

25. $(49a^4b^6)^{\frac{1}{2}}$.

26. $\sqrt[3]{2a^2b^2}$.

27. $\sqrt{32} \cdot \sqrt{2}$.

28. $(a^2b\sqrt[4]{xy^2})^8$.

29. $(m^3\sqrt[5]{p^4})^4$.

30. $\sqrt{6ab} \cdot \sqrt{2a}$.

31. $[2a(4a^2)^{\frac{1}{3}}]^2$.

32. $11^{\frac{1}{2}} \cdot 11^{\frac{1}{3}} \cdot 11^{\frac{1}{7}}$.

33. $\sqrt[8]{m^2\sqrt[3]{m^2}}$.

34. $\left(\frac{49a^4}{64b^6}\right)^{\frac{1}{2}}$.

35. $\sqrt[3]{a^3\sqrt{25b}}$.

36. $\sqrt{x^2y\sqrt[4]{25yz}}$.

37. $\sqrt[6]{a\sqrt[5]{ab^3}}$.

38. $\sqrt{\sqrt[3]{4x^2}}$.

39. $\sqrt{\sqrt[4]{8x^3}}$.

40. $\sqrt{\sqrt[5]{81a^4}}$.

41. $\sqrt[3]{x^2\sqrt[3]{9a^2}}$.

42. $\sqrt[3]{x^3y^6\sqrt[4]{27a^3}}$.

43. $a(a^2b)^{\frac{1}{3}} \cdot b(ab^2)^{\frac{1}{3}}$.

44. $3\sqrt{9a} \cdot 3^{\frac{1}{2}}$.

45. $5\sqrt[4]{a^3x^3} \cdot 2\sqrt[3]{ax}$.

46. $3\sqrt{8} \cdot 2\sqrt[3]{6} \cdot 3\sqrt[4]{54}$.

MEANING OF ZERO AND NEGATIVE EXPONENTS

323. The meaning of a^0 may be found as follows:

Assuming that Law I holds for a^0, $\quad a^5 \cdot a^0 = a^{5+0}$

$$= a^5.$$

Dividing by a^5, $\quad a^0 = \frac{a^5}{a^5} = 1.$

That is:

Any number (not zero) with the exponent zero equals 1.

Thus, $5^0 = 1$, $10^0 = 1$, $(\frac{10}{3})^0 = 1$, $\left(\frac{a}{b}\right)^0 = 1$.

ORAL EXERCISES

State the value of:

1. $(ab)^0$.
2. $.9^0$.
3. 100^0.
4. $\left(\dfrac{a}{bc}\right)^0$.
5. $(-3)^0$.
6. $(\frac{1}{2})^0$.
7. $(-\frac{1}{27})^0$.
8. $3^2 \cdot 3^0$.
9. $3^2 \cdot 5^0$.
10. $\frac{1}{2} \cdot a^0$.
11. a^5a^0.
12. $a^m \cdot b^0$.
13. $a^5 + a^0$.
14. $4^3 + 4^0$.
15. $(\frac{1}{2})^0$.
16. $.9 + 100^0$.
17. $9\frac{1}{2} \cdot 5^0$.
18. $3^0 \cdot 27^0$.
19. $(2^3 \cdot 3^2)^0$.

324. The meaning of the negative exponent may be found as follows:

Assuming that Law I holds for negative exponents,

$$5^{-3} \cdot 5^{+3} = 5^{-3+3} = 5^0 = 1.$$

That is, 5^{-3} is a multiplier such that its product with 5^{+3} is 1. But if the product of two numbers is 1, one is the reciprocal of the other.

Therefore, 5^{-3} is the reciprocal of 5^3 which is $\frac{1}{5^3}$.

Expressed in general terms:

$$a^{-n} \cdot a^n = a^{-n+n}$$
$$= a^0 = 1.$$
$$a^{-n} = \frac{1}{a^n}.$$

In words:

a^{-n} *means* $\frac{1}{a^n}$, *for all values of* n, *positive or negative, integral or fractional.*

ORAL EXERCISES

Find similarly the meaning of:

1. 4^{-3}.
2. 2^{-4}.
3. $(\frac{3}{4})^{-2}$.
4. $(-5)^{-6}$.
5. a^{-3}.
6. $a^{-\frac{3}{4}}$.
7. $(\frac{1}{2})^{-\frac{2}{3}}$.
8. $(-\frac{2}{3})^{-\frac{4}{5}}$.

State the value of:

9. $4^{-\frac{1}{2}}$.
10. $8^{-\frac{1}{3}}$.
11. $8^{-\frac{2}{3}}$.
12. $(.2^0)^{\frac{1}{2}}$.
13. $16^{-\frac{1}{4}}$.
14. $16^{-\frac{3}{4}}$.
15. $.125^{-\frac{1}{3}}$.
16. $.125^{-\frac{2}{3}}$.
17. $32^{-\frac{3}{5}}$.
18. $27^{-\frac{2}{3}}$.
19. $.36^{-\frac{3}{2}}$.
20. $64^{-\frac{5}{6}}$.
21. a^{-3}.
22. $(a^0)^{-2}$.
23. $a^{-\frac{2}{5}}$.
24. $a^{-\frac{1}{3}}$.

WRITTEN EXERCISES

Perform the operations indicated:

1. $(4^2 \cdot 5^4)^{-\frac{1}{2}}$.
2. $2^{14} \cdot 2^{-6}$.
3. $2^{\frac{2}{3}} \cdot 2^{-\frac{5}{6}}$.
4. $2^{-\frac{2}{3}} \cdot 2^{\frac{5}{6}}$.
5. $2^{-\frac{2}{3}} \cdot 2^{-\frac{5}{6}}$.
6. $3^{\frac{7}{6}} \cdot 3^{-\frac{1}{2}} \cdot 4^0$.
7. $\frac{5^{-\frac{1}{2}} \cdot 5^{\frac{3}{2}}}{5^0}$.
8. $10^{-5} \cdot 10^2 \cdot 10^0$.
9. $10^7 \cdot 10^{-7}$.
10. $27^{-\frac{3}{4}} \cdot 9^{\frac{3}{2}}$.
11. $\sqrt[4]{18^{-4}} \cdot \sqrt[3]{18^{-3}}$.
12. $\sqrt[3]{10^{-4}} \cdot \sqrt[5]{10^0}$.
13. $(10^{-4} \cdot 10^{-8})^{\frac{2}{7}}$.
14. $10^{-\frac{2}{3}} \cdot 10^{\frac{1}{2}}$.
15. $10^{\frac{1}{2}} \cdot 10^{\frac{5}{6}} \cdot 10^{\frac{1}{3}} \cdot (\frac{4}{7})^0$.
16. $(x^{-4})^2$.
17. $a^0 \cdot a^{\frac{1}{3}}$.
18. $\frac{a^{-\frac{1}{2}}}{a^{-\frac{1}{3}}}$ (or $a^{-\frac{1}{2}-(-\frac{1}{3})}$).
19. $a^{\frac{1}{2}} \cdot a^{-\frac{1}{2}}$ (or $a^{\frac{1}{2}-\frac{1}{2}}$).
20. $\frac{a^{\frac{1}{2}}}{a^{-\frac{1}{2}}}$ (or $a^{\frac{1}{2}-(-\frac{1}{2})}$).
21. $(a^{-\frac{1}{2}})^{-\frac{1}{3}}$.
22. $(x^{-4})^5$.

USE OF ZERO, NEGATIVE, AND FRACTIONAL EXPONENTS

325. We have defined zero and negative exponents so that Law I holds for them. It can be shown that the other four laws hold for these exponents as defined, but the laws will be applied here without proof.

326. The relation $a^{-n} \cdot a^n = a^0 = 1$ can be used to change the form of expressions.

I. **To free an expression from a negative exponent,** *multiply both numerator and denominator by a factor that will so combine with the factor having the negative exponent as to produce unity in accordance with the relation just mentioned. If more than one negative exponent is involved, apply the process for each.*

For example:

$$7^{-3} \cdot 2^4 = \frac{7^3 \cdot 7^{-3} \cdot 2^4}{7^3} = \frac{2^4}{7^3}.$$

$$\frac{a^3}{b^{-5}} = \frac{b^5 \cdot a^3}{b^5 \cdot b^{-5}} = \frac{b^5a^3}{1} = b^5a^3.$$

$$\frac{x^{-2}}{t^{-5}} = \frac{t^5x^2 \cdot x^{-2}}{t^5x^2 \cdot t^{-5}} = \frac{t^5}{x^2}.$$

WRITTEN EXERCISES

Free from negative exponents:

1. $\dfrac{3^{-2}}{2^{-3}}$.
2. $\dfrac{a^{-5}}{b^{-3}}$.
3. $\dfrac{3^{-2} \cdot 4^{-1}}{5^{-3}}$.
4. $\dfrac{a^{-1}b^{-2}}{x^{-4}}$.
5. $\dfrac{x^{-a}}{a^{-x}}$.
6. $\dfrac{a^{-3}}{b^{3}}$.
7. $2\,a^{-\frac{1}{4}}$.
8. $\dfrac{ab}{c^{-5}}$.
9. $a^{-1}b^{-2}$.
10. $\dfrac{ax^{-4}}{b^{-3}x^{2}}$.
11. $\dfrac{5\,y^{3}}{5^{-1}y^{-4}}$.
12. $\dfrac{1}{a^{-3}x^{-3}}$.

II. **To free an expression from a fractional form,** *multiply both numerator and denominator by a factor that, in combination with the denominator, will produce unity. If more than one such form is involved, apply the process for each.*

For example:

$$\frac{a^3}{b^2} = \frac{b^{-2}a^3}{b^{-2}b^2} = b^{-2}a^3;$$

also,

$$\frac{2}{4^{-5}} + \frac{a^3}{x^2y^{-3}} = \frac{4^5 \cdot 2}{4^5 \cdot 4^{-5}} + \frac{x^{-2}y^3a^3}{x^{-2}y^3x^2y^{-3}}$$

$$= 4^5 \cdot 2 + \frac{x^{-2}y^3a^3}{(x^{-2}x^2)(y^3y^{-3})}$$

$$= 2 \cdot 4^5 + a^3x^{-2}y^3.$$

WRITTEN EXERCISES

Free from fractional forms:

1. $\dfrac{3}{5^2}$.
2. $\dfrac{1}{a^2b^3}$.
3. $\dfrac{a}{bx^3}$.
4. $\dfrac{a^{-2}}{t^{-3}}$.
5. $\dfrac{1}{5^2} + \dfrac{3}{2^3}$.
6. $\dfrac{x^2}{y^3} + \dfrac{y^2}{x^3}$.
7. $\dfrac{x}{y^{-1}} + \dfrac{a^2}{x^{-3}}$.
8. $\dfrac{a}{b} + \dfrac{a^{-1}}{b^{-1}}$.

III. **To transfer any specified factor from the numerator into the denominator, or vice versa,** *multiply the numerator and denominator by a factor that, in combination with the factor to be transferred, will produce unity.*

EXAMPLES

1. Transferring the factors of the denominator to the numerator:

$$\frac{x^7}{x^4y^{-3}} = \frac{x^{-4}y^3x^7}{x^{-4}y^3x^4y^{-3}} = x^{-4}y^3x^7 = x^3y^3.$$

2. Transferring the literal factors of the numerator to the denominator:

$$\frac{5\,a^3b^2}{4\,a^4b^{-3}} = \frac{5\,a^{-3}b^{-2}a^3b^2}{4\,a^{-3}b^{-2}a^4b^{-3}} = \frac{5}{4\,ab^{-5}}.$$

WRITTEN EXERCISES

In the following expressions:

(*a*) Transfer all literal factors to the numerator.

(*b*) Transfer all literal factors to the denominator.

1. $\dfrac{a^2x^3}{a^4x^7}$.

2. $\dfrac{6\,ay^2z^{-5}}{11\,a^3x^{-3}y^4}$.

3. $\dfrac{4\,a^2b^3c^5}{17\,a^3b^5c^{-3}}$.

4. $\dfrac{5\,a^{-\frac{1}{3}}b^{\frac{2}{5}}}{8\,a^{\frac{5}{6}}b^{-\frac{3}{5}}}$.

5. $\dfrac{a^{-\frac{2}{3}}b^{-5}c^{\frac{3}{4}}}{a^{\frac{1}{3}}b^{-5}c^{-\frac{3}{4}}}$.

6. $\dfrac{7(a^{-7}b^{-5})^{\frac{1}{4}}}{6\,a^{-\frac{7}{4}}b^{-\frac{3}{2}}}$.

7. $\dfrac{5\,a^{-\frac{1}{3}}}{abx}$.

8. $\dfrac{6\,x^{-\frac{4}{5}}y^2}{5\,pq}$.

9. $\dfrac{a^{\frac{3}{4}}}{b^{\frac{7}{8}}}$.

327. The laws of exponents enable us to perform operations with polynomials containing fractional and negative exponents.

Thus:

$$(a^{\frac{2}{3}} + b^{\frac{1}{2}})^2 = (a^{\frac{2}{3}})^2 + 2\,a^{\frac{2}{3}}b^{\frac{1}{2}} + (b^{\frac{1}{2}})^2$$

$$= a^{\frac{4}{3}} + 2\,a^{\frac{2}{3}}b^{\frac{1}{2}} + b.$$

WRITTEN EXERCISES

Perform the indicated operations:

1. $(a^{\frac{3}{2}} - b^{\frac{5}{4}})^2$.
2. $(4\,a^{\frac{2}{3}}x^{\frac{1}{2}} + 2^3)^2$.
3. $(a^{-\frac{1}{2}} + b^{\frac{3}{2}})^2$.
4. $(a^n - 3\,b^r)^2$.
5. $(x^{-\frac{3}{4}} - y^{-\frac{2}{3}})(x^{-\frac{3}{4}} + y^{-\frac{2}{3}})$.
6. $(x^{-n} + 1)(x^{-n} - 1)$.
7. $(x^{\frac{6}{5}} + 3)(x^{\frac{6}{5}} + 5)$.
8. $\dfrac{a^5 - x^4}{a^{\frac{5}{2}} - x^2}$.
9. $(x^n - y^n)^2$.
10. $(a^{2n}b^{3r} - 1)^2$.
11. $\dfrac{a^p \cdot b^q}{a^{p+1} \cdot b^{q-1}}$.
12. $(a^n + t^n)^2$.
13. $(a^{\frac{1}{3}}b^{\frac{3}{2}} + x^{-\frac{5}{2}})(a^{\frac{1}{3}}b^{\frac{3}{2}} - x^{-\frac{5}{2}})$.

Express as a product of two factors:

14. $x^6 - m^{-4}$.
15. $a^8 - 2\,a^4x^{\frac{1}{3}} + x^{\frac{2}{3}}$.
16. $a^{2n} + 2\,a^nb^n + b^{2n}$.
17. $x^{4n} - 4\,x^{2n}y^{2n} + 4\,y^{4n}$.
18. $y^{-7} - x^{-10}$.
19. $x^r - 4$.
20. $1 + 8\,x^{-\frac{5}{2}} + 16\,x^{-5}$.
21. $x^{12} + 6\,x^6y^{-\frac{7}{2}} + 9\,y^{-7}$.

SUMMARY

I. **Definitions.**

Meaning of the Fractional Exponents. $a^{\frac{p}{q}}$ denotes the qth root of the pth power, or the pth power of the qth root of a. Sec. 315.

Meaning of the Exponent Zero. Any number (not 0) with the exponent zero equals 1. Sec. 323.

Meaning of Negative Exponents. a^{-n} means $\dfrac{1}{a^n}$ for all values of n, positive or negative, integral or fractional. Sec. 324.

II. **Laws of Exponents.**

1. $a^m \cdot a^r = a^{m+r}$. Sec. 307.

2. $a^m \div a^r = a^{m-r}$, if $m>r$. Sec. 308.

3. $(a^m)^r = a^{mr}$. Sec. 309.

4. $(ab)^n = a^n b^n$. Sec. 309.

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$. Sec. 309.

These laws apply for all values of the exponents m, n, r, positive, negative, integral, or fractional.

III. **Processes with Exponents.**

1. When the bases of the factors are the same, the exponent of the product is found by adding the given exponents (Law I). Sec. 317.

2. When the bases of the expressions are the same, the exponent of the quotient is found by subtracting the exponent of the divisor from the exponent of the dividend (Law II). Sec. 318.

3. When an exponent is applied to a number having an exponent, the product of the exponents is taken as the exponent of the result (Law III). Sec. 319.

4. An exponent affecting a product is applied to each factor (Law IV). Sec. 320.

5. An exponent affecting a fraction is applied to both numerator and denominator (Law V). Sec. 320.

6. When the bases are different and the fractional exponents are different, the exponents must have, or be made to have, a common denominator, before any simplification by multiplication or division is possible. Sec. 321.

7. To free an expression from a negative exponent, multiply both numerator and denominator by a factor that will so combine with the factor having the negative exponent as to produce unity, in accordance with the relation $a^{-n} \cdot a^n = a^0$. Sec. 326.

8. To free an expression from the fractional form, multiply numerator and denominator by a factor that, in combination with the denominator, will produce unity. If more than one such form is involved, apply the process for each. Sec. 326.

9. To transfer any specified factor from the numerator into the denominator, or vice versa, multiply the numerator and denominator by a factor that, in combination with the factor to be transferred, will produce unity. Sec. 326.

REVIEW

WRITTEN EXERCISES

Express with positive exponents:

1. $m^{-\frac{3}{4}}n^8$. 2. $4\,x^{-\frac{1}{2}}y^{-1}z$. 3. $3\,a^{-5}b^5$. 4. $17\,x^{-\frac{5}{2}}y^{-7}z^{-\frac{7}{8}}$.

Transfer all literal factors from the denominator to the numerator:

5. $\dfrac{x^{\frac{3}{4}}}{x^{-\frac{2}{3}}}$. 6. $\dfrac{ab}{a^{-\frac{1}{2}}b^{-3}}$. 7. $\dfrac{1}{6\,x^{-2}y^{\frac{3}{2}}}$. 8. $\dfrac{5}{ay^{-4}}$.

Multiply:

9. $(2+\sqrt{x+1})^2$.

10. $\sqrt{5}\cdot\sqrt[3]{6}$.

11. $5\sqrt[4]{m^{-3}}\cdot 2m^{-1}$.

12. $p\cdot p^{-\frac{3}{4}}$.

13. $(a^{-1}-b^{-1})(a^{-\frac{1}{2}}-b^{-\frac{1}{2}})$.

14. $(x^2-1)(x^{\frac{1}{2}}+1)$.

Remove the parentheses:

15. $(a^{-\frac{3}{4}})^2$.

16. $(\sqrt[3]{a^{-1}})^3$.

17. $\left(\dfrac{1}{\sqrt[4]{p^3}}\right)^{\frac{4}{3}}$.

18. $\left(\dfrac{1}{n^{-2}}\right)^5$.

19. $\left(a^{\frac{5n}{6}}\right)^{\frac{3}{10n}}$.

20. $\left(x^{\frac{m^2}{n^2}-1}\right)^{\frac{n}{m-n}}$.

21. $\left(\dfrac{1}{x^{-\frac{p}{q}}}\right)^{\frac{q}{p}}$.

22. $\left(m^{\frac{pq}{xy}}\right)^{\frac{x^2y}{pq}}$.

23. $\left(s^{\frac{1}{n-1}}\right)^{n^2-1}$.

SUPPLEMENTARY WORK

ADDITIONAL EXERCISES

Perform the operations indicated:

1. $(8\,a^3b^{\frac{1}{2}}c^{-\frac{3}{4}})^{\frac{2}{3}}$.

2. $(16\,a^{\frac{2}{3}}bc^{-4})^{\frac{3}{4}}$.

3. $(5\,ax^3y^2)^{\frac{1}{2}}$.

4. $(\sqrt[3]{4\,x^2})^{\frac{1}{4}}$.

5. $(64\,x^{-3}y^{\frac{1}{4}})^{-\frac{1}{6}}$.

6. $(-250\,x^3y^{-\frac{3}{5}}z^{-3})^{\frac{2}{3}}$.

7. $(-243\,a^{-\frac{1}{2}}y^{-1}z^{\frac{1}{4}})^{-\frac{4}{5}}$.

8. $\sqrt[p-q]{x^{p^2-q^2}}$.

9. $\sqrt[p]{x^{-2p}y^{-p^2}z^{-p}}$.

10. $\sqrt{a^{-m}b^{4m}}$.

11. $3\,a^{\frac{1}{2}}b^{\frac{2}{3}}c^{-1}\cdot 2\,a^{\frac{1}{4}}b^{\frac{1}{2}}c$.

12. $10\,ab^{-\frac{1}{2}}c^{\frac{3}{4}} \div 2\,a^{-1}b^2z^{\frac{1}{2}}$.

13. $a^mb^{-2n}c^{-2} \div ab^nz^{-3}$.

14. $(3\,a^{-2} \div b^{-2})^{-5}$.

15. $(a^{p-q})^{p+q}\cdot a^{p^2}a^{q^2}$.

16. $\dfrac{\sqrt{a}}{\sqrt[5]{a^2b^3}}\cdot\dfrac{\sqrt[4]{a^3b^2}}{\sqrt[10]{a^7b^9}}$.

17. $x^{-3}\cdot x^{\frac{3}{4}}\cdot x^2\cdot\sqrt[5]{x}$.

18. $\dfrac{\sqrt[3]{x}}{\sqrt{x}}\cdot\dfrac{x^3}{x}\cdot\dfrac{\sqrt[3]{x^2}}{x^2}$.

19. $[\{(a^2-b^{-2})^{-1}\}^{-2}]^5$.

20. $[(a^{-\frac{1}{2}})^{\frac{2}{3}}]^{-12}$.

21. $\left\{\sqrt{ab^{-2}\sqrt{ab}}\right\}^4$.

22. $\{(a^{-3}b^2)^{\frac{1}{2}}\}^{-\frac{2}{3}}$.

23. $\sqrt[3]{a^2\sqrt{a^{-1}}}$.

24. $[(x^a)^{-b}]^{-\frac{1}{a}} \div [(x^{-b})^c]^{-\frac{1}{c}}$.

25. $-a^3b^{-4}c^{-3}d^5\cdot -a^{-2}b^5c^4d^{-5}$.

26. $a^2x-3\,y^6\cdot a^3x^6y^9$.

27. $3\,x^{-\frac{1}{2}}5\,x^{\frac{4}{5}}\cdot 10\,x^{-\frac{1}{2}}$.

28. $\sqrt{a^{-1}\sqrt{a^3\sqrt{a-4}}}$.

29. $\{x^{-\frac{3}{2}}y(xy^{-2})^{-\frac{1}{2}}(x^{-1}y)^{\frac{2}{3}}\}^{\frac{3}{2}}$.

Simplify:

30. $$\dfrac{\dfrac{x-y}{2\,y+x}+\dfrac{1}{2}}{3\frac{1}{2}-\left(\dfrac{x+2\,y}{5\,x+7\,y}\right)^{-1}}.$$

31. $\left(x^{\frac{n+1}{n-1}} \div x^{\frac{n-1}{n+1}}\right)^{\frac{n-1}{2n}}$.

32. $[(a^{p+q})^{p-q}(a^{q^2})^{\frac{1}{q}}]^{\frac{1}{p^2}}$.

33. (a) $\dfrac{\sqrt{(2^{\frac{1}{3}}+2^{-\frac{1}{3}})^2-4}}{4(2^{\frac{1}{3}}-2^{-\frac{1}{3}})}\cdot\dfrac{5+\sqrt{21}}{5-\sqrt{21}}$.

(b) $\sqrt[7]{x^2y^{12}}\left(\dfrac{1}{xy}\right)^{\frac{1}{7}}\left(\dfrac{y^2}{x}\right)^{-\frac{2}{7}}$.

(c) $3\sqrt{\dfrac{5}{2}}+\sqrt{40}+\sqrt{\dfrac{2}{5}}-\dfrac{1}{\sqrt{10}}$.

34. $\frac{1}{3}\sqrt{45}+4\sqrt{\frac{5}{4}}-\sqrt{125}$.

35. Express the product $\sqrt[3]{a^2}\sqrt{a^3}$ as a single radical.

36. Divide $2x^{\frac{9}{2}}y^{-3}-5x^{\frac{7}{2}}y^{-2}+7x^{\frac{5}{2}}y^{-1}-5x^{\frac{3}{2}}+2x^{\frac{1}{2}}y$ by $x^3y^{-3}-x^2y^{-2}+xy^{-1}$.

Find equivalent expressions with rational divisors:

37. $3\sqrt{2a}\div 2\sqrt{3b}$.

38. $b\sqrt{a^2}\div\sqrt{ab}$.

39. $\sqrt{40x^3y}\div x\sqrt{5y}$.

40. $x\sqrt[3]{y}\div y\sqrt[3]{x}$.

41. $2\sqrt[3]{2a^2}\div\sqrt[3]{4a}$.

42. $-\frac{3}{4}\sqrt{\frac{3}{5}}\div\frac{3}{10}\sqrt{3}$.

43. $6\sqrt[3]{54x^2}\div 2\sqrt[3]{2x^2}$.

44. $a^2\sqrt[4]{48ab^3}\div 2ab\sqrt[4]{3ab^2}$.

45. $4ax\div\sqrt[4]{ax}$.

46. $\dfrac{\sqrt{a+b}+\sqrt{a-b}}{\sqrt{a+b}-\sqrt{a-b}}$.

47. $\dfrac{1}{a-\sqrt{a^2-x^2}}$.

48. $\dfrac{(\sqrt{5}-2)(3+\sqrt{5})}{5-\sqrt{5}}$.

49. $\dfrac{p-\sqrt{q}}{p+\sqrt{q}}$.

50. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}$.

51. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$.

52. $\dfrac{\sqrt{x}-\sqrt{x+y}}{\sqrt{x}+\sqrt{x+y}}$.

53. $\dfrac{\sqrt{a^2+b^2}+b}{\sqrt{a^2+b^2}-b}$.

54. $\dfrac{3\sqrt{x-3}+\sqrt{x+3}}{3\sqrt{x-3}-\sqrt{x+3}}$.

55. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$.

56. $\dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}$.

57. $\dfrac{1}{2+\sqrt{5}-\sqrt{2}}$.

Suggestion. Rationalize the denominator in two steps, using as first factor

$$2-(\sqrt{5}-\sqrt{2}).$$

Solve:

58. $3x+\sqrt{x^2-2x+5}=1$.

59. $\dfrac{\sqrt{x}+a}{b}=\dfrac{x}{a^2-2ab+b^2}$.

60. $\dfrac{x}{a+b}+\sqrt{(a+b)^2+ab-x}=a+b-\dfrac{2ab}{a+b}$.

61. $\sqrt{x}(2a-b+\sqrt{x})=3a^2-ab$.

CHAPTER XIX

INVOLUTION AND EVOLUTION

INVOLUTION

328. The operation of raising an expression to a given power is called **involution**.

An important case of involution arises when the given expression is a binomial.

329. PREPARATORY.

1. We have already found that

$$(a+b)^2 = a^2 + 2\,ab + b^2$$
$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

The method of multiplication by detached coefficients (p. 75) enables us to write out readily the coefficients of $(a+b)^4$. For

$$(a+b)^4 = (a+b)(a+b)^3.$$

$$\begin{array}{ccccc} 1+3+3+1 \\ 1+1 \\ \hline 1 & 3 & 3 & 1 & \\ & 1 & 3 & 3 & 1 \\ \hline 1 & 4 & 6 & 4 & 1 \end{array}$$

$$\therefore (a+b)^4 = a^4 + 4a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$

2. From this find similarly $(a+b)^5$.

3. From $(a+b)^5$ find similarly $(a+b)^6$.

330. The result of multiplying out a power of a binomial is called a **binomial expansion**.

331. The coefficients of the successive powers of a binomial may be arranged in the form of a triangular table:

Expansions	Coefficients
$(a+b)^0 = 1$	1
$(a+b)^1 = a+b$	1 1
$(a+b)^2 = a^2 + 2\,ab + b^2$	1 2 1
$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3$	1 3 3 1
$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$	1 4 6 4 1

Supply the next two lines, using the results of Exercises 2 and 3, Sec. 329.

The table on the right is called **Pascal's triangle,** and the numbers the **binomial coefficients.**

Each number of Pascal's triangle is the sum of the number directly above it and the number to the left of that.

That this must be so follows from the process of multiplying by $a + b$, and is readily seen when the method of detached coefficients is used. The table enables us easily to write expansions of successive powers of $a + b$.

332. Any expression that can be put into the form of a binomial expansion may be written as a power of a binomial by inspection.

For example :

1. $16\,x^4 - 32\,x^3y + 24\,x^2y^2 - 8\,xy^3 + y^4$
$= (2\,x)^4 - 4 \cdot 8\,x^3y + 6 \cdot 4\,x^2y^2 - 4 \cdot 2\,xy^3 + y^4$
$= (2\,x)^4 + 4 \cdot (2\,x)^3(-y) + 6(2\,x)^2(-y)^2 + 4(2\,x)(-y)^3 + (-y)^4$
$= (2\,x - y)^4.$

2. $a^3 + 3\,a^2b + 3\,ab^2 + b^3 - 3\,a^2c - 6\,abc - 3\,b^2c + 3\,ac^2 + 3\,bc^2 - c^3$
$= (a+b)^3 + 3(a+b)^2(-c) + 3(a+b)(-c)^2 + (-c)^3$
$= (\overline{a+b} - c)^3.$

ORAL EXERCISES

Express as a power of a binomial:

1. $-a^3 + 3\,a^2 - 3\,a + 1.$
2. $a^{12} + 3\,a^8 + 3\,a^4 + 1.$
3. $1 - 3\,x^4 + 3\,x^8 - x^{12}.$
4. $x^6 - 3\,x^4y^2 + 3\,x^2y^4 - y^6.$
5. $8\,a^3 - 12\,a^2b + 6\,ab^2 - b^3.$
6. $x^4 - 4\,x^3y + 6\,x^2y^2 - 4\,xy^3 + y^4.$

7. $x^4 - 4x^3 + 6x^2 - 4x + 1.$

8. $(2x)^4 - 4(2x)^3 + 6(2x)^2 - 4(2x) + 1.$

9. $16x^4 + 32x^3 + 24x^2 + 8x + 1.$

10. $16x^4 - 32x^3 + 24x^2 - 8x + 1.$

11. $a^5 - 5a^4 + 10a^3 - 10a^2 + 5a - 1.$

12. $a^2 + 2ab + b^2 - 2ac - 2bc + c^2.$

13. $(2a)^4 + 4(2a)^3 + 6(2a)^2 + 4(2a) + 1.$

14. $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$

15. $(a-b)^4 - 4(a-b)^3c + 6(a-b)^2c^2 - 4(a-b)c^3 + c^4.$

16. $x^6 - 6x^5(2y) + 15x^4(2y)^2 - 20x^3(2y)^3 + 15x^2(2y)^4 - 6x(2y)^5 + (2y)^6.$

17. State in order the coefficients in the expansion of a binomial of the fourth degree. Of the third degree. Of the fifth degree.

333. The Binomial Formula. It can be proved that

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}a^{n-4}b^4 + \cdots.$$

This is known as the **binomial formula.**

The factor 1 is understood in each denominator; and by denoting the product $1\cdot 2\cdot 3$ by $3!$ (read "three factorial"), and generally $1\cdot 2\cdot 3 \cdots k$ by $k!$, the above formula can be written:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \cdots.$$

The law according to which the terms are written is as follows: the factorial number in the denominator is the exponent of b; the exponent of a is n diminished by that of b; the numerator of the coefficient is a product of factors beginning with n, decreasing successively by 1, and ending with that in which n is diminished by one less than the factorial number in the denominator.

The symbol $\underline{|k}$ is also used with the same meaning as $k!$

WRITTEN EXERCISES

1. Test the binomial formula just given for $n=3$.

It should, of course, reduce to the known expression for $(a+b)^3$. The formula comes to an end because the factor $n-3$ occurs in all the terms after a certain one, and when $n=3$, this factor is zero.

2. Test similarly for $n=4, 5, 6, 2, 1$.

Expand:

3. $(x+y)^4$.	7. $(x-1)^5$.	11. $(ab-cd)^4$.
4. $(x-y)^4$.	8. $(x-2y)^6$.	12. $(t-u)^6$.
5. $(x-y)^5$.	9. $(ab+1)^4$.	13. $(3a+b)^7$.
6. $(x+y)^5$.	10. $(bc-1)^4$.	14. $(x+5)^5$.

15. Write the next two terms of the binomial formula as given in Sec. 350.

16. Observe that if the successive terms were written according to the same law, the *tenth* term of the binomial formula would be, $\frac{n(n-1)(n-2)\ \cdots\ (n-8)}{9!}a^{n-9}b^9$. Write the fifteenth term.

17. By reference to the binomial formula, state the number of the last term written in the following expression:

$$(x+2y)^8 = x^8 + 8x^7(2y) + \frac{8\cdot 7}{2!}x^6(2y)^2 + \cdots + \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{6!}x^2(2y)^6 + \cdots.$$

Write the first three terms and the seventh term in the expansion of each of the following:

18. $(x+y)^{12}$.	20. $(ab-y)^{11}$.	22. $(2x-1)^9$.
19. $(1+5x)^{14}$.	21. $\left(\frac{x}{2}+\frac{y}{3}\right)^{12}$.	23. $(4x+3)^{20}$.

24. Reduce the terms written in Exercise 21 to their simplest form.

25. If the expansions in Exercises 18–23 were written out in full, how many terms would each have?

26. In each of Exercises 18–23, determine whether or not there is a middle term in the expansion. If there is a middle term:

(*a*) Determine its number.

(*b*) Write it out, without simplifying.

(*c*) Simplify the result.

EVOLUTION

334. The process of extracting an indicated root is called **evolution.** The most important case of evolution is the extraction of square root.

All the numbers that we have hitherto considered, whether positive or negative, have positive squares; none of them has a negative square, consequently the square root of a negative number (as, for example, $\sqrt{-5}$) has no meaning at this stage of our work, but will be explained later (Sec. 346, p. 314).

335. PREPARATORY.

1. $[\pm(m+n)]^2=?$ Hence, $\sqrt{m^2+2\,mn+n^2}=?$
2. $[\pm(m+2\,n)]^2=?$ Hence, $\sqrt{m^2+4\,mn+4\,n^2}=?$
3. $[\pm(a-3)]^2=?$ Hence, $\sqrt{a^2-6\,a+9}=?$

336. Square Root by Inspection. The formula

$$[\pm(a+b)]^2=a^2+2\,ab+b^2$$

shows that when one of the terms of the trinomial is twice the product of the square roots of the other two, the trinomial is the square of the sum of these square roots. By aid of this relation the square roots of certain trinomials can be found readily by inspection.

For example:

$\sqrt{4\,a^2-12\,ab+9\,b^2}=\pm\,(2\,a-3\,b)$, since $-12\,ab$ is twice the product of the square roots of $4\,a^2$ and $9\,b^2$. These roots must be taken with opposite signs in this case because twice their product is to be negative.

ORAL EXERCISES

Find the square root of:

1. $x^2 - 2x + 1$.
2. $x^2 + 4x + 4$.
3. $x^2 - 2xy + y^2$.
4. $4x^2 + 8x + 4$.
5. $a^2b^2 + 2ab + 1$.
6. $a^2 - 4ab + 4b^2$.
7. $4m^2 - 4mn + n^2$.
8. $a^2b^2 + 2abc + c^2$.
9. $x^4 - 4x^2 + 4$.
10. $16x^8 + 8x^4 + 1$.
11. $4a^2b^2 - 4ab + 1$.
12. $a^6 + 2 + a^{-6}$.
13. $x + 2 + x^{-1}$.
14. $x^{4a} + 2x^{2a}y^b + y^{2b}$.
15. $a^{8x} - 6a^{4x}b^{\frac{y}{2}} + 9b^y$.
16. $a^{-6} - 2a^{-3}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.
17. $25a^4 - 10a^2 + 1$.
18. $25a^2 - 30ab + 9b^2$.
19. $49a^2b^2 - 14a^3b + a^4$.
20. $16x^2y^2 + 40xy^2z + 25y^2z^2$.
21. $25a^4b^2c^2 + 10a^2bc^5 + c^8$.
22. $4t^{\frac{4}{7}} - 20p^{\frac{3}{2}}t^{\frac{2}{7}} + 25p^3$.

337. Square Roots of Arithmetical Numbers. The square root of arithmetical numbers can be found approximately by inspection.

EXAMPLES

1. Find approximately $\sqrt{19}$.

19 lies between 16 and 25.
Therefore, $\sqrt{19}$ lies between $\sqrt{16}$ and $\sqrt{25}$, or between 4 and 5.
That is, $\sqrt{19}$ is 4 plus a decimal.

2. Find approximately $\sqrt{643}$.

643 lies between 400 and 900.
Therefore, $\sqrt{643}$ lies between $\sqrt{400}$ and $\sqrt{900}$, or between 20 and 30.
That is, it is 20 plus a number less than 10.

3. Find approximately $\sqrt{4678}$.

4678 lies between 3600 and 4900.
Therefore, $\sqrt{4678}$ lies between $\sqrt{3600}$ and $\sqrt{4900}$, or between 60 and 70.
That is, it is 60 plus a number less than 10.

The numbers to be added in any case will not change the first figure of the root found. That is, *by inspection we can find exactly the first figure of the square root.*

ORAL EXERCISES

State the first figure of the square root of each number:

1. 24.	5. 219.	9. 1247.	13. 8500.
2. 51.	6. 317.	10. 1721.	14. 6274.
3. 73.	7. 843.	11. 4693.	15. 9200.
4. 96.	8. 999.	12. 4000.	16. 53,249.

338. Pointing off into Periods. Since $10^2 = 100$, we know that the square root of any number greater than 1 but less than 100 is less than 10. Its integral part consists of one figure.

Since $100^2 = 10,000$, we know that the square root of any number greater than 100 but less than 10,000 is greater than 10 but less than 100.

That is, if the given number has 3 or 4 digits in its integral part, its square root will have 2 digits in its integral part. If larger numbers are given, the above reasoning can be repeated for 1000^2, etc., showing that in all cases if the number be pointed off into periods of 2 digits each (or possibly fewer in the left period), then each period will correspond to a digit of the root.

Thus in 67′62′31, there are three periods, therefore there are three places in the integral part of the root. Since 67 lies between 64 and 81, the square root of 67 is approximately 8, and that of 676,231 is approximately 800.

ORAL EXERCISES

By the method above state an approximate square root of:

1. 12′36.	6. 1′25′00.	11. 43′21′00.
2. 30′95.	7. 8′23′00.	12. 58′61′23.
3. 45′18.	8. 8′39′99.	13. 58′94′55.
4. 85′00.	9. 67′50′00.	14. 99′93′33.
5. 50′00.	10. 75′00′00.	15. 83′46′25.

339. When the first digit of the square root has been found by inspection, the process may be continued thus:

EXAMPLE

Find $\sqrt{2209}$:

1. The approximate value, as above, is 40.
2. Let $\sqrt{2209} = 40 + r$, where r is less than 10.
3. $\therefore 2209 = (40 + r)^2 = 40^2 + 2 \cdot 40 \cdot r + r^2$.
4. $\therefore 2209 - 40^2 = 2 \cdot 40 \cdot r + r^2$, or $609 = 80\,r + r^2$.
5. Then 609 is greater than $80\,r$, or $\frac{609}{80}$ is greater than r.
6. But $\frac{609}{80} = 7 +$ decimal.

$\therefore 7 +$ decimal is greater than r, and it is possible that $7 = r$.

Trying, we find that $80 \cdot 7 + 7^2 = 609$.

That is, $r = 7$.

Therefore, $\sqrt{2209} = 40 + 7 = 47$.

WRITTEN EXERCISES

Use the above process to find the square roots of:

1. 625.	6. 441.	11. 1225.	16. 9801.
2. 169.	7. 1024.	12. 3025.	17. 7225.
3. 361.	8. 8464.	13. 7744.	18. 9025.
4. 256.	9. 4225.	14. 5625.	19. 6889.
5. 6724.	10. 1521.	15. 2025.	20. 8281.

340. When once an approximate value, a, has been found for the root, an approximate value for the remainder, r, of the root can be found by means of the formula: $(a + r)^2 = a^2 + 2ar + r^2$.

1. Let n denote the number whose square root is sought, a denote the approximate root at any stage, and r the remainder of the root.
2. Then, $n = (a + r)^2 = a^2 + 2\,ar + r^2$.
3. $\therefore\ n - a^2 = 2\,ar + r^2$.
4. Or, $n - a^2$ is greater than $2\,ar$, or, $\dfrac{n - a^2}{2\,a}$ is greater than r.
5. Hence, $\dfrac{n - a^2}{2\,a}$ may be tried as an approximate value of r.

341. What precedes may be formulated into a process or working rule, thus:

1. *Point off the number into periods of two figures each, beginning at units' place (at the decimal point).*

2. *By inspection find the largest integer whose square is not greater than the left period.* (In Example A it is 9.)

(A)

Root	9 2
Number	84'64
	81
18	364
182	364

3. *Use this integer as the first digit of the root. Subtract its square from the left period.* (In Example A this square is 81.)

4. *Bring down the next period.* (In Example A this makes 364.)

5. *Multiply the part of the root already found by 2. This number is called the* **trial divisor**. (18 in Example A.)

6. *Divide the remainder (omitting the right digit) by the trial divisor and use the digit found as the next digit of the root.*
(In Example A, $36 \div 18 = 2$.)

(B)

Root	3 0. 6 9
Number	9'41'.87'61
	9
6	41
60	4187
606	3636
612	55161
6129	55161

7. *Annex this digit to the trial divisor. This forms the* **complete divisor**. (182 in A.)

8. *Multiply the complete divisor by the digit of the root just found and subtract.*

Note. It may happen that the product to be subtracted is larger than the number from which it is to be subtracted. This indicates that the trial divisor produced too large a digit. Try the next smaller digit for the figure of the root last found.

(C)

Root	1 4. 1 4+
Number	2'00.
	1
2	100
24	96
28	400
281	281
282	11900
2824	11296
	604

9. *Repeat the steps 4 to 8 until all of the periods have been brought down.*

If the last remainder is zero, as in Example B, the process is ended, the given number is a perfect square, and its root has been found exactly. If the last remainder is not zero, as in C, the process may be continued as far as desired by supplying zeros.

Test. The square of the root, if complete, equals the given number.

342. When a number contains a decimal the decimal point of its root is placed between the figures furnished by the integral periods and those furnished by the decimal periods (as in C p. 320).

WRITTEN EXERCISES

Find the square roots:

1. 361.	3. 625.	5. 2025.	7. 177,241.	9. 4,334,724.
2. 784.	4. 841.	6. 1936.	8. 120,409.	10. 4,888,521.

Find the square roots to two decimal places:

11. 2.25.	14. 19.36.	17. 2000.	20. 3.
12. 7.84.	15. 90.25.	18. 0.03.	21. 111.
13. 6.25.	16. 1.21.	19. 5.	22. 0.00111.

343. To find the square roots of fractional numbers, *either first reduce the fraction to a decimal or extract the square root of both numerator and denominator.*

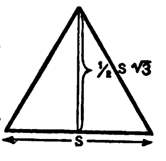
WRITTEN EXERCISES

Find the square roots:

1. $\frac{81}{144}$.	3. $\frac{256}{441}$.	5. $\frac{625}{3025}$.	7. $\frac{622521}{724201}$.
2. $\frac{169}{225}$.	4. $\frac{361}{841}$.	6. $\frac{289}{961}$.	8. $\frac{7695076}{7871225}$.

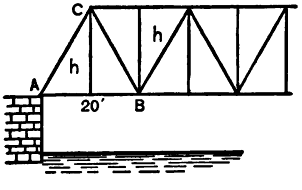
9. The area of an equilateral triangle of side s is known to be $\frac{1}{4}s^2\sqrt{3}$. Find the side of an equilateral triangle whose area is $9\sqrt{3}$ sq. in.

SUGGESTION: Solve the equation $\frac{1}{4}s^2\sqrt{3} = 9\sqrt{3}$ for s.



10. The altitude of an equilateral triangle is $\frac{1}{2}s\sqrt{3}$. Find the side of an equilateral triangle whose altitude is $128\sqrt{3}$ ft.

11. Each section, as *ABC*, of the bridge truss shown in the picture is an equilateral triangle. Find the height, *h*, of the bridge.



12. If the side of an equilateral triangle is 10 ft., find the altitude within .01 ft.

14. It is known that if the sides of a parallelepiped are a, b, and c, the diagonal, d, is $\sqrt{a^2+b^2+c^2}$. If the sides of a parallelepiped are 20 yd., 30 yd., and 50 yd., find its diagonal within .001 yd.

15. It is known that the volume of a frustum of a pyramid of altitude h and of bases with areas b_1 and b_2 is $\frac{h}{3}(b_1+b_2+\sqrt{b_1b_2})$. Find the volume (within .01 cu. ft.) of a frustum whose bases are 20 sq. ft. and 90 sq. ft. and whose altitude is 15 ft.

16. Compute $\sqrt{s(s-a)(s-b)(s-c)}$ within .01 in., when $s=18.5$ in., $a=10$ in., $b=15$ in., $c=12$ in.

Solve and compute the values of the roots to three figures:

17. $9x+\dfrac{2}{x-1}=21.$ 18. $80x-16.1x^2=21.$

344. Square Roots of Polynomials. The square root of every polynomial that is a square may be extracted according to the process given in Sec. 341, p. 304.

EXAMPLE

Extract the square root of $a^2-2ab+b^2-2ac+2bc+c^2$.

$$
\begin{array}{ll}
\textbf{Root} & \underline{a-b-c.} \\
\textbf{Power} & a^2-2ab-2ac+b^2+2bc+c^2 \\
 & \underline{a^2-2ab \qquad\quad +b^2} \\
 & \qquad\quad -2ac \qquad +2bc+c^2 \\
 & \qquad\quad \underline{-2ac \qquad +2bc+c^2}
\end{array}
$$

1. As far as possible, arrange the terms according to the descending powers of some letter, as a in this case.

2. The square root of the first term is the first term of the root. (Corresponds to steps 2, 3, Sec. 341.)

3. Divide the second term of the power by twice the first term of the root, $2a$ in this case. The result is the second term of the root. (Steps 5, 6.)

4. Subtract from the power the square of the binomial found.

5. If there is a remainder (as $-2ac+2bc+c^2$ in this case) it shows that the power contains the square of a trinomial and that there is at least another term in the root.

This term (c) is found by dividing the remainder by twice the part of the root found [$2(a-b)$ in this case], for the same reason as in the square root of numbers. (Step 6.)

6. The square of the entire root so far found $(a-b-c)^2$ must now be subtracted. We have already subtracted the square of the first part of the new binomial $[(a-b)^2$ in this case$]$. Therefore, subtract the rest of the square $[-2(a-b)c+c^2$, or $-2ac+2bc+c^2]$.

Briefly, the trial divisor, $2(a-b)$, is augmented by the next term, $-c$, resulting in $2(a-b)-c$, and this is multiplied by $-c$. This gives the part $-2(a-b)c+c^2$, still to be subtracted. This is analogous to what is done in extracting the square roots of numbers. (Steps 7, 8, Sec. 341.)

If there is a new remainder, divide it by twice the entire part found and proceed as before.

Test. Square the root. The result should be the power.

WRITTEN EXERCISES

Extract the square root of:

1. $49a^2b^2-14a^3b+a^4$.
2. $16x^2y^2+40xy^2z+25y^2z^2$.
3. $4x^4+4x^3+5x^2+2x+1$.
4. $x^4-6x^3+11x^2-6x+1$.
5. $1+4x+10x^2+12x^3+9x^4$.
6. $9x^4+12x^3+22x^2+12x+9$.
7. $9a^2+12ab+4b^2+6ac+4bc+c^2$.
8. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$.
9. $x^6-4x^5+6x^4+2x^3-11x^2+6x+9$.

345. The process of the preceding section may be used to approximate the square root of any polynomial.

Thus the square root of $1-x^2$ to three terms is found as follows:

$$\text{Root } 1-\frac{x^2}{2}-\frac{x^4}{8}$$

$$\text{Power } 1-x^2$$

$$\begin{array}{r|l}
1^2; & 1 \\ \hline
 & -x^2 \\
2\left(-\frac{x^2}{2}\right)+\frac{x^4}{4}; & -x^2+\frac{x^4}{4};\ \left(1-\frac{x^2}{2}\right)^2 \text{ has now been subtracted.} \\ \hline
 & -\frac{x^4}{4} \\
2\left(1-\frac{x^2}{2}\right)\left(-\frac{x^4}{8}\right)+\frac{x^8}{64}; & -\frac{x^4}{4}+\frac{x^6}{8}+\frac{x^8}{64};\ \left(1-\frac{x^2}{2}-\frac{x^4}{8}\right)^2 \text{ has now been subtracted.} \\ \hline
 & -\frac{x^6}{8}-\frac{x^8}{64}
\end{array}$$

Test. The square of the root found plus the remainder should be the given expression.

WRITTEN EXERCISES

Find the square root of each expression to three terms:

1. $1 - x$.
2. $4 + x$.
3. $a^2 + b$.
4. $x^2 + xy$.
5. $1 - 2a$.
6. $16 + mn$.
7. $16a^2 + 12ab$.
8. $9m^2 + 9mn$.

SUMMARY

1. Raising numbers to powers is called *involution*, and extracting roots is called *evolution*. Secs. 328 and 334.

2. A *binomial expansion* is the result of multiplying out a power of a binomial. Sec. 330.

3. Any expression which can be put into the form of a binomial expansion can be expressed as a power of a binomial by inspection. Sec. 332.

4. The *binomial formula* is:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \cdots.$$ Sec. 333.

5. The square roots of trinomials of the form $a^2 + 2ab + b^2$ can be found by inspection. Sec. 336.

6. The general process for finding square roots is based upon the formula $(a+r)^2 = a^2 + 2ar + r^2$. Sec. 340.

7. To extract the square root of a fraction, first reduce the fraction to a decimal, or extract the square root of both numerator and denominator. Sec. 343.

REVIEW

WRITTEN EXERCISES

Extract the square root of:

1. 14,641.
2. 1.5625.
3. $a^4 + 2a^2x + x^2$.
4. $9a^4 + 12a^3 - 20a^2 - 16a + 16$.
5. $a^2 + b^2 + 4c^2 - 2ab + 4ac - 4bc$.
6. $m^4 + 4am^3 + 6a^2m^2 + 4a^3m + a^4$.

7. $\frac{y^4}{16} - \frac{y^3}{4z} + \frac{3y^2}{20z^2} + \frac{y}{5z^3} + \frac{1}{25z^4}$.

8. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

Expand by the binomial formula:

9. $(a+\frac{1}{2})^7$.

10. $(ax+1)^5$.

11. $(mx^2-1)^6$.

12. $(a^{\frac{1}{3}}+5)^4$.

13. $(ax^2-2y)^7$.

14. $(3x^2-\frac{1}{2})^5$.

15. $\left(\frac{a}{b}-\frac{b}{a}\right)^4$.

16. $\left(\frac{1}{2x^2}-\frac{1}{3y^2}\right)^5$.

SUPPLEMENTARY WORK

Cube Root of Arithmetical Numbers.

Pointing off into Periods. Since $10^3 = 1000$, we know that the cube root of any number greater than 1 but less than 1000 is less than 10. Its integral part consists of one figure.

Since $\overline{100}^3 = 1{,}000{,}000$, we know that the cube root of any number greater than 1000 but less than 1,000,000 is greater than 10 but less than 100. That is, if the given number has from 4 to 6 digits in its integral part, its cube root will have 2 digits in its integral part. If larger numbers are given, the above reasoning can be repeated for 1000^3, etc., showing that in all cases if the number be pointed off into periods of 3 digits each (or possibly fewer in the left period), then each period will correspond to a digit of the root.

The cube root of the left period can be found approximately by inspection, and the number so found, with a zero annexed for each other period, will be an approximate value for the root.

EXAMPLE

Find $\sqrt[3]{481890304}$.

Pointing off as above:

$$481'890'304.$$

By trial we find $7^3 = 343$, and $8^3 = 512$. Hence 700 is an approximate value for the root. It may be verified that $\overline{700}^3$ is less than the given number, and that $\overline{800}^3$ is more than the given number. That is, the hundreds' figure of the root is 7.

WRITTEN EXERCISES

Find, as above, the first figure of:

1. $\sqrt[3]{493039}$.
2. $\sqrt[3]{2924207}$.
3. $\sqrt[3]{57066625}$.
4. $\sqrt[3]{254840104}$.
5. $\sqrt[3]{1345572864}$.
6. $\sqrt[3]{62287505344}$.

When once an approximate value a has been found for the root, an approximate value for the remainder, r, of the root can be found by means of the formula:

$$(a+r)^3 = a^3 + 3a^2r + 3ar^2 + r^3.$$

For example:

In $\sqrt[3]{238328}$ we find as above that $a = 60$.

$$
\begin{array}{lrl}
\text{Then,} & a^3 + 3a^2r + 3ar^2 + r^3 = & 238'328, \text{ the whole cube,} \\
\text{Subtracting} & a^3 \qquad\qquad\qquad = & 216\,000, \text{ the cube of the part found,} \\
\hline
& 3a^2r + 3ar^2 + r^3 = & 22'328, \text{ the first remainder.}
\end{array}
$$

Since something must be added to $3a^2r$ to make it equal to 22,328,

$$3a^2r \text{ is less than } 22'328,$$

or
$$r \text{ is less than } \frac{22'328}{3a^2}.$$

Consequently, the first figure of this quotient will either be the first figure of r or greater than it. In this instance $3a^2$ is 10,800, hence the first figure of the quotient is 2.

Trying 2 as r, we have to calculate $3a^2r + 3ar^2 + r^3$. This is most conveniently done by using the form $(3a^2 + 3ar + r^2)r$.

$$
\begin{array}{lr}
\text{We have already} & 3a^2 = 10800 \\
\text{We find:} & 3ar = \quad 360 \\
 & r^2 = \qquad 4 \\
\hline
\text{Adding:} & 3a^2 + 3ar + r^2 = 11164 \\
\text{Then,} & r(3a^2 + 3ar + r^2) = 22328
\end{array}
$$

The calculation should be arranged thus:

$$
\begin{array}{llr|l}
 & & & 6 \quad 2 \\
\cline{4-4}
 & & & 238'328 \\
 & & a^3 = & 216\,000 \\
\cline{3-4}
\text{Trial divisor:} & & 3a^2 = 10800 & \quad 22328 \\
 & 3ar = 3 \cdot 2 \cdot 60 = & 360 & \\
 & & 2^2 = \qquad 4 & \\
\cline{2-3}
\text{Complete divisor:} & & 11164 & \quad 22328 \text{ or } 2 \cdot 11164 \\
\cline{4-4}
\end{array}
$$

If the root consists of more than two figures, the above work is repeated, using the part of the root already found as a.

If it should happen that the product of r and the complete divisor is larger than the remainder of the number, try the next smaller digit for r.

When necessary, periods are pointed off to the right of the decimal point.

WRITTEN EXERCISES

Find:

1. $\sqrt[3]{74088}$.
2. $\sqrt[3]{148877}$.
3. $\sqrt[3]{2803221}$.
4. $\sqrt[3]{16387064}$.
5. $\sqrt[3]{55306341}$.
6. $\sqrt[3]{143055667}$.

Find to one decimal place:

7. $\sqrt[3]{637}$. 8. $\sqrt[3]{3485}$. 9. $\sqrt[3]{263488}$.

Find to two decimal places:

10. $\sqrt[3]{5}$. 11. $\sqrt[3]{17}$. 12. $\sqrt[3]{269}$. 13. $\sqrt[3]{4.763}$.

Cube Root of Polynomials

The *cube roots* of polynomials may be extracted in a similar manner:

EXAMPLE

Extract the cube root of $27x^3 - 27x^2y + 9xy^2 - y^3$.

$$
\begin{array}{lr}
 & \text{Root} \quad 3x - y \\
a^3 + 3a^2r + 3ar^2 + r^3 = \text{Power} & \overline{27x^3 - 27x^2y + 9xy^2 - y^3} \\
a = 3x \qquad\qquad a^3 = (3x)^3 = & 27x^3 \qquad\qquad\qquad\qquad \\
\text{Trial divisor} = 27x^2 \quad \therefore r = -y & \overline{-27x^2y + 9xy^2 - y^3} \\
\text{Complete divisor} = [3(3x)^2 + 3(3x)y + y^2] & \underline{-27x^2y + 9xy^2 - y^3}
\end{array}
$$

EXPLANATION.

1. Arrange the expression in the order of the powers of some letter, as x.

2. Take the cube root of the first term of the power for the first term of the root, as $3x$.

3. Divide the second term of the power by 3 times the square of the first term of the root. The result is the second term of the root, as $-y$.

4. If a denote the approximate value of the root already found ($3x$ in the above instance), and r the value to be used as the next term ($-y$ in the above instance), form the complete divisor $3a^2 + 3ar + r^2$, multiply it by r, and subtract.

5. If there is a further remainder, proceed as before, using the entire part of the root already found as a.

WRITTEN EXERCISES

Extract the cube root of:

1. $8x^3 + 12x^2 + 6x + 1$.
2. $27 - 27x + 9x^2 - x^3$.
3. $8a^3b^3 - 12a^2b^2 + 6ab - 1$.
4. $x^3 - 3x^2y + 3xy^2 - y^3$.
5. $8x^6 - 12x^4 + 6x^2 - 1$.
6. $x^{12} - 3x^8m + 3x^4m^2 - m^3$.

ADDITIONAL EXERCISES

1. Find the middle term of the expansion of $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$.

2. Raise 98 to the 5th power by the binomial theorem.

SUGGESTION. Use $100-2$ for 98.

3. Solve the equation $(x+2)^7-(x-2)^7=2^8$.

SUGGESTION. Expand by binomial theorem, indicating the powers of 2. Perform the subtraction and in the result put $x^2=y$, obtaining a cubic equation in y with y as a factor. Either $y=0$, or the quadratic factor $=0$. Find the value of x under each supposition.

4. Find, in its simplest form, the 13th term of $(1-x)^{17}$, when

$$x=\frac{a^{-\frac{1}{4}}(a-b)^{\frac{9}{4}}}{\sqrt[3]{1-\frac{b}{a}}}\times\frac{1}{(b-a)^2}.$$

5. Find in its simplest form the middle term of the expansion of

$$\left(\frac{2^{-\frac{1}{5}}19^{-\frac{2}{5}}}{\sqrt[20]{121}}-\frac{17^{-\frac{1}{10}}}{19^{-\frac{3}{10}}}\right)^{20}.$$

6. Find the ratio between the 6th term in the expansion of $\left(\frac{1+3x}{2}\right)^{10}$ and the 5th term in the expansion of $\left(\frac{1+3x}{2}\right)^{9}$.

CHAPTER XX

IMAGINARY AND COMPLEX NUMBERS

346. Imaginary Numbers. The numbers defined in what precedes have all had positive squares. Consequently, among them the equation $x^2 = -3$, which asks, "What is the number whose square is -3?" has no solution.

A solution is provided by defining a new number, $\sqrt{-3}$, as a number whose square is -3. Similarly we define $\sqrt{-a}$, where a denotes a positive number, as a number whose square is $-a$.

The square roots of negative numbers are called **imaginary numbers.**

347. If a is positive, $\sqrt{-a}$ may be expressed $\sqrt{a}\ \sqrt{-1}$.

Similarly,
$$\sqrt{-5} = \sqrt{5(-1)} = \sqrt{5}\ \sqrt{-1}.$$
$$\sqrt{-49} = \sqrt{49(-1)} = 7\sqrt{-1}.$$

348. Real Numbers. In distinction from imaginary numbers, the numbers hitherto studied are called **real numbers.**

WRITTEN EXERCISES

Express as in Sec. 347:

1. $\sqrt{-9}$.	4. $\sqrt{-100}$.	7. $\sqrt{-18}$.	10. $\sqrt{-12}$.
2. $\sqrt{-16}$.	5. $-\sqrt{-64}$.	8. $-\sqrt{-32}$.	11. $\sqrt{-50}$.
3. $\sqrt{-25}$.	6. $\sqrt{-8}$.	9. $-\sqrt{-7}$.	12. $-\sqrt{-75}$.

349. The positive square root of -1 is frequently denoted by the symbol i; that is, $\sqrt{-1} = i$.

Using this we write:

$$\sqrt{-5} = \sqrt{5} \cdot i.$$
$$\sqrt{-49} = \pm 7i.$$
$$\sqrt{-75a^2b} = \sqrt{3 \cdot 25a^2b \cdot -1} = 5a\sqrt{3b} \cdot i.$$

NOTE. Throughout this chapter the radical sign is taken to mean the positive root only.

WRITTEN EXERCISES

Rewrite the following, using the symbol i as in Sec. 349:

1. $2+\sqrt{-4}$.
2. $3-\sqrt{-9}$.
3. $4+\sqrt{-4}$.
4. $5-\sqrt{-16}$.
5. $25-\sqrt{-25}$.
6. $5-\sqrt{-3}$.
7. $3+\sqrt{-6}$.
8. $7+\sqrt{-12}$.
9. $12-\sqrt{-9}$.
10. $2\sqrt{-100}$.
11. $4\sqrt{-(a+b)}$.
12. $\sqrt{a}+\sqrt{-b^2c^2}$.
13. $-\sqrt{-b^2c}$.
14. $a+\sqrt{-(a^2+x^2)}$.
15. $x+y-\sqrt{-xy^2}$.
16. $p^2+\sqrt{-(p+q)^3}$.

350. Complex Numbers. A binomial one of whose terms is real and the other imaginary is called a **complex number.**

The general form of a complex number is $a+bi$, where a and b may be any real numbers.

NOTE. Complex numbers are also simply called **imaginary**, any expression which involves i being called imaginary. Single terms in which i is a factor (those which we have called imaginary above) are often called **pure imaginaries**, while the others are called **complex imaginaries**. Thus, $\sqrt{-2}$, $3\sqrt{-a}$, $5i$ are pure imaginaries and $1-\sqrt{-3}$, $a-\sqrt{-b}$ are complex imaginaries.

ORAL EXERCISES

1. Name the real term and the imaginary term in each exercise of the last set.

2. Name the values of a and b in each exercise.

351. Processes with Imaginary and Complex Numbers. After introducing the symbol i for the imaginary unit $\sqrt{-1}$, the operations with imaginary and complex numbers are performed like the operations with real numbers.

I. *Addition and Subtraction.*

EXAMPLE

Add $\sqrt{-9}$, $-\sqrt{-25}$, $\sqrt{-3}$.

$$\sqrt{-9} = 3i.$$
$$-\sqrt{-25} = -5i.$$
$$\sqrt{-3} = -\sqrt{3}\cdot i.$$

$\therefore$ the sum is $(3-5-\sqrt{3})i = -(2+\sqrt{3})i.$

WRITTEN EXERCISES

Add:

1. $2i, 3i, -i.$
2. $\sqrt{16}i, -2i.$
3. $\sqrt{-16}, -2\sqrt{-1}.$
4. $\sqrt{-4}, \sqrt{-9}, \sqrt{-1}.$
5. $6-\sqrt{-5}, 2\sqrt{-4}, \sqrt{-25}.$
6. $3+4i, 2-3i, 5+5i.$
7. $\sqrt{-9x^2}, -\sqrt{-8x^2}.$
8. $\sqrt{-(a+b)^2}, -\sqrt{(b+c)^2}.$
9. $2\sqrt{-32a^3}, 3\sqrt{-8a^3}, 6\sqrt{2}i.$
10. $\sqrt{3}i-1, \sqrt{2}i+2, i-2\sqrt{2}.$

II. *Multiplication.*

To multiply complex numbers we apply the fact that $\sqrt{-1}\cdot\sqrt{-1}=-1$, or $i^2=-1$, since the square of the square root of a number is the number itself.

EXAMPLES

Multiply:

1. $\sqrt{-16}$ by $\sqrt{-9}.$

$$\begin{array}{l} \sqrt{-16} = 4\sqrt{-1} = 4i. \\ \underline{\sqrt{-9} \;= 3\sqrt{-1} = 3i.} \end{array}$$

$\therefore$ the product is $12(\sqrt{-1})^2 = (12)(-1) = -12.$

This may be written $(4i)(3i) = 12i^2 = -12.$

2. $3-\sqrt{-3}$ by $2-\sqrt{-5}.$

$$\begin{array}{l} 3-\sqrt{3}i \\ \underline{2-\sqrt{5}i} \\ 6-2\sqrt{3}i \\ \underline{\quad -3\sqrt{5}i+\sqrt{15}i^2} \\ 6-(2\sqrt{3}+3\sqrt{5})i-\sqrt{15}. \end{array}$$

3. $a+bi$ by $a-bi.$

$$\begin{array}{lll} a+bi & & \\ \underline{a-bi} & & \\ a^2+abi & & \\ \underline{\quad -abi} & \underline{-b^2i^2} & \\ a^2 & +b^2. & \end{array}$$

352. $a+bi$ and $a-bi$ are called *conjugate* complex numbers.

WRITTEN EXERCISES

Multiply:

1. $5-3i$ by $5+3i$.
2. $3+\sqrt{-3}$ by $2+\sqrt{-5}$.
3. $5-2\sqrt{-1}$ by $3+2\sqrt{-1}$.
4. $5+\sqrt{-3}$ by $5-\sqrt{-3}$.
5. $3-\sqrt{-2}$ by $3+2\sqrt{-2}$.
6. $1-\sqrt{-7}$ by $2+3\sqrt{-7}$.
7. $4+i$ by $5-i$.
8. $a+xi$ by $a-xi$.
9. a^2+b^2i by a^2-b^2i.
10. $\sqrt{r}+3i$ by $\sqrt{r}-3i$.
11. $\sqrt{-25}$ by $\sqrt{-9}$ by $\sqrt{-5}$.
12. $\sqrt{-a}$ by $\sqrt{-b}$ by $-ci$.

III. *Division.*

Fractions (that is, indicated quotients) may be simplified by rationalizing the denominator (Sec. 302, p. 269).

For example:

1. $\dfrac{\sqrt{-7}}{\sqrt{-5}}=\dfrac{\sqrt{-7}\,\sqrt{-5}}{\sqrt{-5}\,\sqrt{-5}}=\dfrac{\sqrt{7}\cdot\sqrt{5}\,(-1)^2}{-5}=\dfrac{\sqrt{35}}{5}.$

2. $\dfrac{2+\sqrt{-3}}{3+\sqrt{-5}}=\dfrac{(2+\sqrt{-3})(3+\sqrt{-5})}{(3-\sqrt{-5})(3+\sqrt{-5})}=\dfrac{6+3\sqrt{-3}+2\sqrt{-5}-\sqrt{15}}{9-(-5)}$

$$=\frac{1}{14}(6+3\sqrt{-3}+2\sqrt{-5}-\sqrt{15}).$$

3. $\dfrac{x+yi}{x-yi}=\dfrac{(x+yi)^2}{(x-yi)(x+yi)}=\dfrac{x^2+2xyi-y^2}{x^2+y^2}.$

WRITTEN EXERCISES

Write in fractional form and rationalize the denominators:

1. $\sqrt{-6}\div\sqrt{2}$.
2. $1\div(a+xi)$.
3. $\sqrt{-3}\div\sqrt{-5}$.
4. $\sqrt{ax}\div\sqrt{-a}$.
5. $1\div(2-\sqrt{-3})$.
6. $4\sqrt{-1}\div-2\sqrt{-4}$.
7. $a\div(a-bi)$.
8. $(a+bi)\div(a-bi)$.
9. $(3+6i)\div(5+4i)$.
10. $(\sqrt{3}-9i)\div(\sqrt{2}-9i)$.
11. $(x-\sqrt{-7})\div(x+\sqrt{-7})$.
12. $\left(a-\dfrac{\sqrt{-5}}{2}\right)\div\left(a+\dfrac{\sqrt{-5}}{2}\right)$.
13. $(\sqrt{-2}+\sqrt{-5})\div(\sqrt{-5}-\sqrt{-2})$.

353. Powers of the Imaginary Unit. Beginning with $i^2 = -1$ and multiplying successively by i we find:

$$i^2 = -1. \qquad i^6 = i^4 \cdot i^2 = i^2 = -1.$$
$$i^3 = i^2 \cdot i = -i. \qquad i^7 = i^6 \cdot i = -1 \cdot i = -i.$$
$$i^4 = i^2 \cdot i^2 = -1(-1) = +1. \qquad i^8 = i^4 \cdot i^4 = (+1)^2 = +1.$$
$$i^5 = i^4 \cdot i = i. \qquad i^9 = i^8 \cdot i = i.$$

354. By means of the values of i^2, i^3, i^4, any power of i can be shown to be either $\pm i$ or ± 1.

For example: $i^{63} = i^{60} \cdot i^3 = (i^4)^{15} \cdot i^3 = 1^{15} \cdot i^3 = i^3 = -i.$

WRITTEN EXERCISES

Simplify similarly:

1. i^9.	4. i^{16}.	7. i^{54}.	10. i^{143}.
2. i^{10}.	5. i^{21}.	8. i^{56}.	11. i^{400}.
3. i^{12}.	6. i^{27}.	9. i^{198}.	12. i^{3001}.

Perform the operations indicated:

13. $(1+i)^2$

14. $(1-i)^3$.

15. $(1-i)^3 \cdot i^4$.

16. $\left(\dfrac{-1+3i}{2}\right)^3$.

17. $(1+i) \cdot i^6$.

18. $\left(\dfrac{-1-3i}{2}\right)^3$.

19. $(1+i) \cdot (1-i)^2$.

20. $(1+i)^2 \div (1-i)^2$.

IMAGINARIES AS ROOTS OF EQUATIONS

355. Complex numbers often occur as roots of quadratic equations.

EXAMPLE

Solve: $$x^2 + x + 1 = 0. \qquad (1)$$

$$x^2 + x = -1. \qquad (2)$$

Completing the square, $$x^2 + x + \tfrac{1}{4} = \tfrac{1}{4} - 1. \qquad (3)$$

$$\therefore x + \tfrac{1}{2} = \pm\sqrt{-\tfrac{3}{4}}. \qquad (4)$$

$$\therefore x = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3} \cdot i. \qquad (5)$$

TEST: $(-\frac{1}{2} \pm \frac{1}{2}\sqrt{3} \cdot i)^2 + (-\frac{1}{2} \pm \frac{1}{2}\sqrt{3} \cdot i) + 1 = 0.$

WRITTEN EXERCISES

Solve and test, expressing the imaginary roots in the form $a+bi$:

1. $x^2+5=0$.
2. $x^2+2x+2=0$.
3. $x^2-x+1=0$.
4. $x^2+x+5=0$.
5. $x^2+2x+37=0$.
6. $x^2-8x+25=0$.
7. $x^2-6x+10=0$.
8. $m^2+4m+85=0$.
9. $x^2+10x+41=0$.
10. $x^2+30x+234=0$.
11. $y^2-4y+53=0$.
12. $z^2-6z+90=0$.
13. $p^2+20p+104=0$.
14. $2x^2+4x+3=0$.
15. $3x^2+2x+1=0$.
16. $12t^2+24=0$.
17. $6w^2+30=0$.
18. $8t^2+t+6=0$.
19. $7x^2+x+5=0$.
20. $6x^2+3x+1=0$.
21. $4x^2+4x+3=0$.
22. $12x^2+x+1=0$.
23. $8v^2+3v+6=0$.
24. $w^2+5w+6=0$.
25. $9z^2+2z+5=0$.
26. $7x^2-3x+4=0$.
27. $15z^2+5z-1=0$.
28. $16x^2-8x+1=0$.
29. $10x^2-2x+3=0$.
30. $7t^2-t+1=0$.

356. The occurrence of imaginary roots in solving equations derived from problems often indicates the impossibility of the given conditions.

EXAMPLE

A rectangular room is twice as long as it is wide; if its length is increased by 20 ft. and its width diminished by 2 ft., its area is doubled. Find its dimensions.

SOLUTION. 1. Let $x=$ the width of the room, and $2x$ its length.

2. Then $(2x+20)(x-2)=2\cdot 2x\cdot x$, or $x^2-8x+20=0$.

3. Solving (2), $x=4\pm 2i$.

The fact that the results are complex numbers shows that no actual room can satisfy the conditions of the problem.

WRITTEN EXERCISES

Solve and determine whether or not the problems are possible:

1. In remodeling a house a room 16 ft. square is changed by lengthening one dimension a certain number of feet and by diminishing the other by twice that number. The area of the resultant room is 260 sq. ft.; what are its dimensions?

2. A triangle has an altitude 2 in. greater than its base, and an area of 32 sq. ft.; find the length of its base.

3. A train moving x mi. per hour travels 90 mi. in $15-x$ hours. What is its rate per hour?

SUMMARY

I. **Definitions.**

1. The square roots of negative numbers are called *imaginary numbers.* Sec. 346.

2. Numbers not imaginary are called *real numbers.* Sec. 348.

3. A *complex number* is a binomial, one of whose terms is a real number and the other an imaginary number. Sec. 350.

II. **Processes.**

1. After introducing the symbol i for the imaginary unit $\sqrt{-1}$, the operations with imaginary and complex numbers are performed like the operations with real numbers. Sec. 351.

2. Any power of i can be expressed by $\pm i$ or ± 1. Sec. 354.

3. The solution of quadratic equations may yield complex numbers. In problems this often indicates the impossibility of the given conditions. Secs. 355, 356.

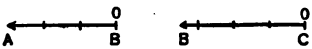
SUPPLEMENTARY WORK

Graphical Representation

We have seen that positive integers and fractions can be represented by lines.

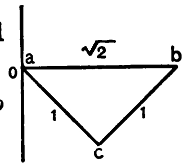
Thus, the line AB represents 3, and the line BC represents $3\frac{1}{2}$.

Similarly, we have seen that negative integers and fractions, which for a long time were considered to be meaningless, can be represented by lines.

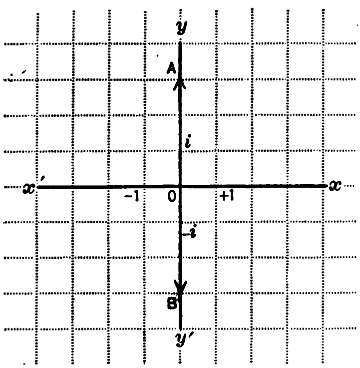


Thus, the line BA represents -3, and the line CB represents $-3\frac{1}{2}$.

Irrational numbers can also be represented by lines.



Thus, in the right-angled triangle abc, the line ab represents the $\sqrt{2}$.



Like the negative number the imaginary number remained uninterpreted several centuries. But this number also can be represented graphically.

Thus, if a unit length on the y-axis be chosen to represent $\sqrt{-1}$ or i, the negative unit $-\sqrt{-1}$ or $-i$ should evidently be laid off in the opposite direction. $3\sqrt{-1}$ or $3i$ would then be represented by OA and $-3i$ by OB, as in the figure, and others similarly.

The reason for placing $\sqrt{-1}$ or i on a line at right angles to the line on which real numbers are plotted may be seen in

the fact that multiplying 1 by $\sqrt{-1}$ twice changes $+1$ into -1. On the graph $+1$ can be changed into -1 by turning it through 180°. If multiplying 1 by $\sqrt{-1}$ twice turns the line 1 through 180°, multiplying 1 by $\sqrt{-1}$ once should turn $+1$ through 90°.

For example:

1. Represent graphically $\sqrt{-4}$:

$\sqrt{-4} = \sqrt{4}\,i = 2\,i$; this is represented by a line 2 spaces long drawn upward on the y-axis.

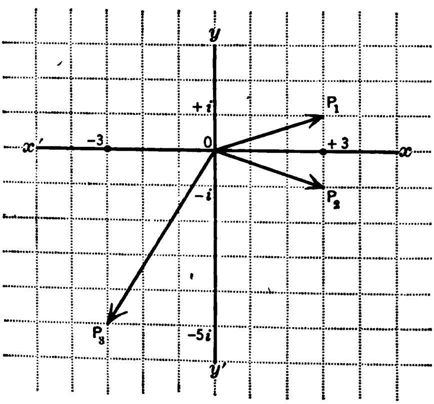
2. Represent graphically $-\sqrt{-3}$:

$-\sqrt{-3} = -\sqrt{3}\,i = -1.7\,i$ (approximately); this is represented by a line 1.7 i spaces long drawn downward on the y-axis.

WRITTEN EXERCISES

Represent graphically:

1. $3\,i$.
2. $-2\,i$.
3. $\sqrt{-9}$.
4. $\sqrt{-16}$.
5. $-5\,i$.
6. $5\,i$.
7. $\sqrt{-3}$.
8. $-\sqrt{-12}$.
9. $-5\sqrt{-4}$.
10. $-3\,i$.
11. $+2\sqrt{-3}$.
12. $5\sqrt{-9}$.



Complex numbers may be represented graphically by a modification of the plan used in representing imaginary numbers.

EXAMPLES

1. Represent graphically $3 + i$.

To do this 3 is laid off on the axis of real numbers, (xx'), and i upward on the axis of imaginaries (yy'). As in other graphical work this locates the point P_1 which is taken to represent the complex number, $3 + i$.

The number $\sqrt{a^2+b^2}$ is called the *modulus* of the complex number $a+bi$. As appears from the figure, $OP_1 = \sqrt{3^2+1^2}$, and hence OP_1 represents the modulus of $3+i$.

2. Represent graphically $3-i$.

The point P_2 is the graph of the complex number $3-i$, and OP_2 represents its modulus.

3. Represent graphically $-3-5i$.

The point P_3 is the graph of the complex number $-3-5i$, and OP_3 represents its modulus.

We have thus *interpreted by means of diagrams* positive and negative integers, positive and negative fractions, positive and negative irrational numbers, and positive and negative complex numbers; in fact, all of the numbers used in elementary algebra.

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